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Regular article

Thermodynamic Topology of Einstein-Maxwell-Scalar Black Holes: Insights from Barrow entropy and Logarithmic Corrections

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Abstract. This paper investigates the thermodynamic topology of Einstein-Maxwell-Scalar (EMS) black holes, incorporating logarithmic Barrow entropy corrections. The analysis confirms the robustness of the topological classification, as the system maintains two distinct topological charges, $(\omega = +1, -1)$, yielding a total charge of $W = 0$. The study reinforces the consistency of EMS black hole topological classification with Reissner-Nordström structures.

Keywords: Thermodynamic Topology; Einstein-Maxwell-Scalar Black Holes; Logarithmic Entropy.

Contents

1 Introduction

The study of photon rings and photon spheres in black holes gained momentum with Cunha et al.'s work in 2017 [1,2], where they introduced an approach based on extrema in an adjusted potential function, complemented by the concept of winding numbers. This analytical framework initially focused on flat black holes and was later extended by Wei to encompass asymptotically AdS and dS spacetimes [3]. The subsequent body of research validated the effectiveness of this technique in capturing topological aspects of photon structures. Given that the method fundamentally relied on a scalar energy function, its adaptability to other similarly behaved scalar quantities was evident. Building on this realization, Wei explored its thermodynamic implications and pioneered its application in identifying critical points by analyzing temperature variations. This initial investigation primarily aimed at locating critical states of black holes but did not delve deeply into the nuances of phase transitions. Progressing beyond this foundational work, the methodology was later adapted to the Helmholtz free energy, unveiling valuable insights into black hole phase transitions [4,5]. Over time, extensive studies employing both conventional and topological variants of this free energy-based approach confirmed its utility, leading to a widespread adoption across diverse models. This technique, particularly in its free energy formulation, demonstrated considerable efficacy in representing phase transition characteristics. Motivated by these developments, the other study introduces a novel perspective by redefining the classification parameter traditionally used in phase transition analysis via the Helmholtz free energy. Instead of relying on conventional thermodynamic variables such as temperature or pressure, propose shifting focus toward the central charge and analyzing the associated critical phenomena. The primary objective is to determine whether this alternate characterization offers deeper insights into the topological nature of phase transitions in black holes. Since free energy inherently incorporates entropy, the investigation extends beyond the standard Bekenstein-Hawking entropy framework. By incorporating super statistical entropies, the aim to assess how additional entropy parameters influence phase transition behavior. Despite the fragmented nature of existing studies on super statistics in gravitational systems, a growing body of research underscores the necessity of non-standard statistical treatments in refining our understanding of complex thermodynamic structures. Traditional equilibrium-based statistical mechanics, largely dependent on Boltzmann-type distributions, often fails to adequately capture the thermodynamics of systems experiencing strong fluctuations or non-equilibrium influences. Super statistics, with its ability to account for localized statistical variations, provides a powerful alternative framework for studying these intricate behaviors. Recognizing the relevance of these factors, investigation seeks to clarify the contribution of super statistical formulations in thermodynamic analyses of black holes [3–51]. The structure of this paper is as follows: Section II presents a concise overview of Einstein-Maxwell-Scalar black holes, emphasizing the insights obtained from Barrow logarithmic entropy. Section III introduces the thermodynamic topology framework and applies it to classify phase transitions in our chosen model. Finally, Section IV summarizes the findings and conclusions drawn from this approach.

2 The Model

In the Reissner-Nordström Anti-de Sitter framework, black holes are known to support a trivial scalar field configuration. However, in Einstein–Maxwell-scalar (EMS) theory, a non-minimal interaction with the Maxwell invariant is introduced via the function $K(\phi)$, enabling scalar hair coupling beyond classical no-hair theorems [52–54]. Additionally, gravitational interactions involve electric charge Q and scalar charge $D = -Q^2/(2M)$ [55]. The EMS action is formulated as [56,57]:

$$S = \int d^4x\sqrt{-g}\left(R - 2\nabla^\mu\phi\nabla_\mu\phi - K(\phi)F^2 - V(\phi)\right), \tag{2.1}$$

where $K(\phi)$ modulates the electromagnetic field, $F^2 = F_{\mu\nu}F^{\mu\nu}$ represents the Maxwell invariant, and $V(\phi)$ accounts for the scalar potential. The static, spherically symmetric black hole metric is given by:

$$ds^2 = -u(r)dt^2 + u^{-1}(r)dr^2 + f(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

For the dilaton black hole in a de Sitter spacetime, when $K(\phi) = e^{2\phi}$, the metric functions take the form:

$$u(r) = 1 - \frac{2M}{r} - \frac{1}{3}\lambda f, \quad f(r) = r\left(r - \frac{q^2}{2M}\right), \quad \phi = -\frac{1}{2}\ln\left(1 - \frac{q^2}{Mr}\right). \tag{2.3}$$

which simplifies to:

$$u(r) = 1 - \frac{1}{3}\lambda r\left(r - \frac{q^2}{2M}\right) - \frac{2M}{r}. \tag{2.4}$$

When $q = 0$, this naturally reduces to the Schwarzschild–de Sitter solution. Here, the term λ is proportional to $f(r)$ rather than r^2, which modifies the underlying thermodynamic behavior. Fractal structures have long played a role in physics, dating back to the Koch Snowflake (1904) [58], which introduced a self-similar geometric pattern with a finite area but an infinite perimeter. Sierpinski [59] and Sponge [60] extended these concepts to three-dimensional configurations with finite volume yet diverging surface area. Inspired by these structures, John Barrow [61] proposed a quantum gravitational modification of Schwarzschild black holes, leading to a fractalization of event horizons. In this formulation, N hierarchical substructures emerge with scaling factor $\lambda < 1$, modifying the Schwarzschild radius $R_g = 2GM$. After infinite iterations, the modified volume and area converge as:

$$V_\infty = \frac{4\pi}{3}R_g^3\sum_{n=0}^{\infty}(N\lambda^3)^n, \quad A_\infty = 4\pi R_g^2\sum_{n=0}^{\infty}(N\lambda^2)^n. \tag{2.5}$$

For convergence, the scaling constraint satisfies $\lambda^{-2} < N < \lambda^{-2}$. The entropy under Barrow's statistical treatment takes the form [61–63]:

$$S_B = \left(\frac{A_g}{A_P}\right)^{1+\Delta/2}, \tag{2.6}$$

where $A_g = 4\pi R_g^2$ represents the Schwarzschild black hole's surface area, and $A_P \sim 4G$ denotes the Planck area. The parameter Δ (ranging between 0 and 1) encapsulates quantum gravitational corrections without explicitly introducing quantum parameters. The Smarr relation governing the thermodynamic properties of a black hole can be expressed as:

$$M = TS + \Phi Q, \tag{2.7}$$

where M represents the black hole mass, T the temperature, S the entropy, Q the charge, Φ its associated potential. The corresponding first law of thermodynamics for black holes follows:

$$dM = TdS + \Phi dQ. \tag{2.8}$$

Since the Barrow entropy takes the form $S_B = S^{1+\Delta/2}$ and the Hawking temperature is defined as $T = \frac{\partial M}{\partial S}$, a direct relationship between modified thermodynamic variables (T_B, S_B) and their conventional counterparts (T, S) can be formulated:

$$TS = (1 + \Delta/2)T_B S_B. \tag{2.9}$$

By substituting this equation into the Smarr relation, the corresponding expression in terms of modified quantities becomes:

$$M = (1 + \Delta/2)T_B S_B + \Phi Q. \tag{2.10}$$

This outcome deviates from the conventional Smarr relation, where the mass parameter exhibits homogeneity of order one across thermodynamic variables. The presence of quantum gravitational effects, captured by the fractal parameter Δ, disrupts this homogeneity, altering the underlying thermodynamic structure. In the absence of quantum corrections $(\Delta = 0)$, the standard Smarr relation is recovered. Defining the Barrow temperature as $T_B = \frac{\partial M}{\partial S_B}$, one finds:

$$T_B = \frac{\partial M}{\partial S} \cdot \frac{\partial S}{\partial S_B} = T\frac{\partial S}{\partial S_B}. \tag{2.11}$$

which further implies:

$$TdS = T_B dS_B. \tag{2.12}$$

Thus, the first law of thermodynamics for the Barrow-modified black hole is given by:

$$dM = T_B dS_B + \Phi dQ. \tag{2.13}$$

Incorporating Schwarzschild black hole properties, Barrow entropy simplifies as:

$$S_B = (4\pi G)^{(1+\Delta)/2} M^{2+\Delta}. \tag{2.14}$$

Considering gravitational self-interaction and backreaction effects, the entropy is further refined with logarithmic corrections [56,62]:

$$S_{LC} = \left(\frac{A_g}{A_P}\right) + \alpha \ln\left(\frac{A_g}{A_P}\right) + \beta. \tag{2.15}$$

where α, β are constants affecting black hole evaporation rates. Logarithmic modifications alter the entropy scaling, prolonging the evaporation process compared to conventional entropy formulations. This motivates an extended expression incorporating both Barrow entropy and logarithmic corrections [56]:

$$S_{BLC} = \left(\frac{A_g}{A_P}\right)^{1+\delta/2} + \alpha\left(1 + \frac{\delta}{2}\right)\ln\left(\frac{A_g}{A_P}\right) + \beta. \tag{2.16}$$

where δ, α, β are model-dependent parameters. For a Schwarzschild black hole, substituting the appropriate area yields:

$$S_{BLC} = (\pi r_h)^{1+\delta/2} + \alpha\left[\log(2D + r_h) + \left(1 + \frac{\delta}{2}\right)\log(\pi r_h)\right] + \beta. \tag{2.17}$$

From the lapse function condition in Eq. (2.4), the black hole mass as a function of event horizon radius r_h is derived as [56]:

$$M = \frac{1}{12}\left(r_h\sqrt{24\lambda q^2 + \lambda^2 r_h^4 - 6\lambda r_h^2 + 9} - \lambda r_h^3 + 3r_h\right). \tag{2.18}$$

Differentiating the metric function $u(r)$, the Hawking temperature is expressed as:

$$T_H = \frac{1}{4\pi}\left[\frac{2M}{r_h^2} - \frac{1}{3}\lambda\left(r_h - \frac{q^2}{2M}\right) - \frac{\lambda r_h}{3}\right]. \tag{2.19}$$

The thermodynamic volume follows:

$$V = \frac{4\pi}{3}r_h^3. \tag{2.20}$$

Substituting $\lambda = 3P$ in Eq. (2.18), the pressure is obtained as:

$$P = \frac{M(2M - 4\pi r^2 T)}{r^2(2Mr - q^2)}. \tag{2.21}$$

Critical points are determined via the inflection conditions:

$$\frac{\partial P}{\partial V}\Big|_{T=T_c} = 0, \quad \frac{\partial^2 P}{\partial V^2}\Big|_{T=T_c} = 0. \tag{2.22}$$

Solving these yields the critical pressure P_c, volume V_c, and temperature T_c:

$$V_c = \frac{4\pi}{3\left(\frac{3Mr-q^2}{Mr^3}\right)^{3/2}}, \quad T_c = \frac{3Mr - q^2}{2\pi r^3}, \quad P_c = \frac{2M(2Mr + q^2)}{r^3(2Mr - q^2)}. \tag{2.23}$$

3 Thermodynamic topology

The study of black hole thermodynamics has undergone significant transformations, with topological methodologies playing an increasingly crucial role in uncovering intricate phase transition structures beyond conventional thermodynamic potentials. A particularly influential advancement in this regard is Duan's ϕ-mapping theory [4,5]. This framework provides a powerful mechanism for analyzing thermodynamic phase transitions by linking the zeros of a vector field $\boldsymbol{\phi}$ to critical points in the thermodynamic landscape. These singular points act as topological defects, giving rise to a conserved current described by a generalized Jacobian tensor, which simplifies under regular conditions to the determinant of the Jacobian matrix corresponding to the ϕ-mapping. A fundamental topological quantity emerging from this theory is the invariant W, which characterizes global phase structures through a decomposition involving the Hopf index and the Brouwer degree. These mathematical constructs encode the algebraic winding of $\boldsymbol{\phi}$ around its singular points [4,5]. When the topological charge is positive, it signifies a stable black hole phase, whereas a negative value indicates thermodynamic instability, possibly leading to bifurcation phenomena or metastable states. By providing a geometric perspective on gravitational thermodynamics, this approach offers a unifying classification scheme applicable to a wide range of black hole solutions across varying spacetime backgrounds. In this formulation, the thermodynamic phase space is structured around a generalized free energy function constructed from fundamental parameters such as mass and temperature. To satisfy equilibrium conditions, the Euclidean time periodicity τ must correspond to the inverse Hawking temperature T^{-1}, ensuring that the system remains in an on-shell configuration [4,5]. The behavior of free energy is further examined by introducing a vector field $\boldsymbol{\phi}$, whose components represent partial derivatives with respect to horizon parameters or other thermodynamic variables. This methodology has seen widespread application in diverse settings, including black holes in asymptotically

AdS and dS spacetimes, theories incorporating higher-curvature modifications, and cases involving exotic matter interactions. Its adaptability and predictive capabilities make it a powerful tool in modern gravitational thermodynamics. Within this context, the generalized free energy expression is given as:

$$\mathcal{F} = M - \frac{S}{\tau}, \tag{3.1}$$

where τ denotes the Euclidean time period, and its reciprocal corresponds to the temperature of the system. Extensive studies exploring its implications can be found in [3–51]. The free energy remains well-defined only when $\tau = T^{-1}$, ensuring thermodynamic consistency. To further analyze phase structures, the vector field $\boldsymbol{\phi}$ is defined with components:

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot\Theta \csc\Theta \right). \tag{3.2}$$

At the boundary limits $\Theta = 0$ and $\Theta = \pi$, the component ϕ^{Θ} exhibits divergence, with outward vector alignment. The horizon radius r_h extends from 0 to ∞, while Θ varies over $[0, \pi]$. Employing Duan's ϕ-mapping formalism, the topological current satisfies:

$$j^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\epsilon^{ab}\partial_{\nu}n^{a}\partial_{\rho}n^{b}\ , \quad \text{where} \quad n^{a} = \frac{\varphi^{a}}{\|\varphi\|}, \quad \text{where} \quad (\varphi^{1} = \varphi^{r_h}, \varphi^{2} = \varphi^{\Theta}), \tag{3.3}$$

where $\mu, \nu, \rho = 0, 1, 2$. Conservation principles ensure that j^{μ} is nonzero solely at points where $\phi = 0$. The total topological charge W is computed using:

$$W = \int_{\Sigma} j^{0} d^{2}x = \sum_{i=1}^{n} \zeta_{i}\eta_{i} = \sum_{i=1}^{n} \omega_{i}. \tag{3.4}$$

where ζ_i represents the Hopf index, quantifying the winding behavior of $\boldsymbol{\phi}$ around singularities. The term η_i denotes the sign of $j^{0}(\phi/x)_{z_i}$, assuming values ± 1. Finally, ω_i defines the winding number corresponding to the i-th zero within the domain Σ. This section explores the thermodynamic topology of black holes influenced by nonlinear electromagnetic interactions within the presence of a phantom global monopole. Based on Eq. (3.1), the generalized Helmholtz free energy is expressed as:

$$\mathcal{F} = \frac{1}{12} r_h \left(\sqrt{24\lambda q^2 + \left(\lambda r_h^2 - 3\right)^2} - \lambda r_h^2 + 3 \right) - \frac{\beta + \alpha \log(2D + r_h) + \pi^{\frac{\delta}{2}+1} r_h^{\frac{\delta}{2}+1} + \frac{1}{2}\alpha(\delta+2)\log(\pi r_h)}{\tau}. \tag{3.5}$$

Applying Eq. (3.2), the explicit components of the vector field $\boldsymbol{\phi}$ are determined as:

$$\begin{aligned} \phi^{r_h} &= -\frac{\frac{2\alpha}{2D+r_h} + \pi^{\frac{\delta}{2}+1}(\delta+2) r_h^{\delta/2} + \frac{\alpha(\delta+2)}{r_h}}{2\tau} \\ &\quad + \frac{1}{12} r_h \left(\frac{2\lambda r_h \left(\lambda r_h^2 - 3\right)}{\sqrt{24\lambda q^2 + \left(\lambda r_h^2 - 3\right)^2}} - 2\lambda r_h \right) \\ &\quad + \frac{1}{12}\left(\sqrt{24\lambda q^2 + \left(\lambda r_h^2 - 3\right)^2} - \lambda r_h^2 + 3 \right), \\ \phi^{\Theta} &= -\frac{\cot(\Theta)}{\sin(\Theta)}. \end{aligned} \tag{3.6}$$

Solving Eq. (3.6) yields:

$$\tau = \frac{2\sqrt{24\lambda q^2 + (\lambda r_h^2 - 3)^2}\left(2\alpha(\delta+2)D + \pi^{\frac{\delta}{2}+1}(\delta+2)(2D+r_h)r_h^{\frac{\delta}{2}+1} + \alpha(\delta+4)r_h\right)}{r_h(2D+r_h)\left(8\lambda q^2 - (\lambda r_h^2 - 1)\left(\sqrt{24\lambda q^2 + (\lambda r_h^2 - 3)^2} - \lambda r_h^2 + 3\right)\right)}. \tag{3.7}$$

One of the fundamental observations from this analysis is the robustness of the topological classification despite modifications to the free parameters. The system persistently maintains two distinct charges, $(\omega = +1, -1)$, yielding a total topological charge of $W = 0$. This result is evident from Fig.1 and is further substantiated through winding number calculations that confirm the stability of the black hole's phase structure. To better understand the mathematical framework underpinning this classification, we conceptualize the free energy function as a scalar field within a two-dimensional space, parameterized by (r_h, Θ). The corresponding vector field $\boldsymbol{\phi}$ is constructed such that its zero points coincide with extremum values of the free energy function. Depending on whether these points correspond to maxima or minima, the rotational behavior of field lines provides a mechanism for assigning topological charge to each singularity [4]. Fundamental black hole solutions such as the Schwarzschild, Reissner-Nordström, and AdS-Reissner-Nordström configurations exhibit distinctive topological charge classifications: Schwarzschild black hole: $W = -1$, Reissner-Nordström black hole: $W = 0$, AdS-Reissner-Nordström black hole: $W = +1$ [4]. These archetypal solutions establish the basis for categorizing black hole thermodynamics, offering key insights into their stability and evolutionary behavior. Since a black hole with a total charge of $W = 0$ aligns with the characteristics of the Reissner-Nordström family, the findings presented in this study reaffirm that our system consistently conforms to this classification. Across different parameter regimes, the thermodynamic topology remains in agreement with known Reissner-Nordström structures, reinforcing the theoretical framework outlined in [4]. Beyond providing a deeper understanding of stability conditions, this topological formulation offers a systematic approach for analyzing phase transitions in black hole physics. By integrating geometric methods into the study of thermodynamic properties, it establishes a broader connection between gravitational thermodynamics and observable astrophysical phenomena.

4 Conclusion

This study has explored the thermodynamic topology of Einstein-Maxwell-Scalar (EMS) black holes, incorporating logarithmic Barrow entropy corrections to analyze phase structures. A key result is the stability of topological classification, as the system consistently maintains two distinct charges, $(\omega = +1, -1)$, leading to a total topological charge of $W = 0$. This classification remains invariant under modifications of free parameters, highlighting a fundamental consistency in the thermodynamic framework.

By modeling the free energy function as a scalar field in the two-dimensional parameter space (r_h, Θ), the study establishes a direct link between extremum points of free energy and the zero points of the corresponding vector field $\boldsymbol{\phi}$. The rotational characteristics of field lines around these singularities provide a clear method for defining topological charge, distinguishing between stable and metastable configurations.

Fundamental black hole solutions such as Schwarzschild $(W = -1)$, Reissner-Nordström $(W = 0)$, and AdS-Reissner-Nordström $(W = +1)$ serve as reference models, reinforcing the classification scheme. The EMS black hole examined here aligns with the Reissner-

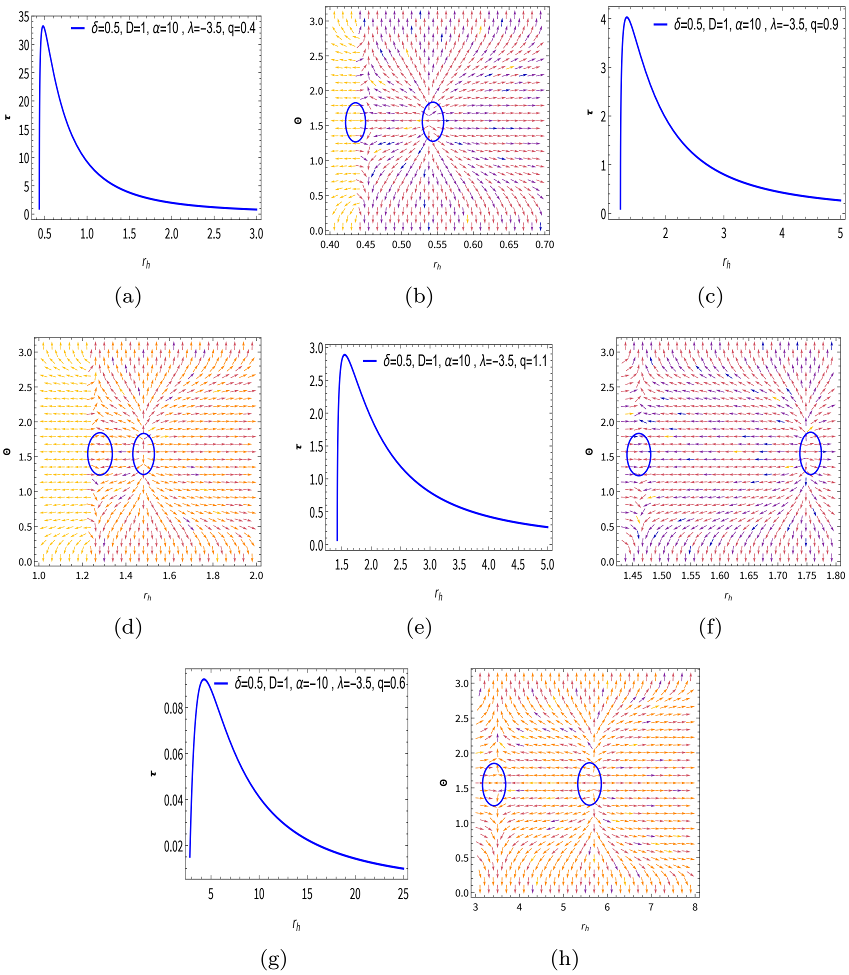

Figure 1: The graphical representation of the relationship between Euclidean time periodicity τ and the horizon radius r_h for the black hole is provided in Figs. (1(a)), (1(c)), (1(e)), (1(g)). These plots serve as a critical tool for examining how thermodynamic variables evolve with respect to horizon radius, particularly in identifying significant phase transition points and stability limits. Additionally, the depiction of the normal vector field n in the (r_h, Θ) plane offers valuable insights into the distribution of Zero Points (ZPs) across specific coordinates. These points correspond to particular choices of free parameters and play an essential role in characterizing thermodynamic stability and topological structures. By analyzing their locations, one can infer fundamental properties of the phase space, including the emergence of distinct thermodynamic behaviors dictated by topological constraints.

Nordström family, reaffirming theoretical expectations and demonstrating consistency across different parameter regimes.

Beyond providing a structured framework for classifying black hole stability, the study extends insights into the nature of phase transitions and thermodynamic evolution. The integration of topological methodologies within gravitational thermodynamics offers a geometric perspective on critical phenomena, enabling deeper connections between black hole physics and astrophysical observations.

This approach broadens the understanding of black hole stability, linking thermodynamic properties with topological invariants. The findings contribute to the ongoing development of topological thermodynamics, providing a systematic strategy for studying black hole phase transitions and enhancing the classification of complex gravitational systems.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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