Journal of Holography Applications in Physics
Volume 5, Issue 1, Winter 2025, 31–49
©Available online at http://jhap.du.ac.ir
DOI: 10.22128/jhap.2024.900.1099
Online ISSN: 2783–3518

Regular article

Redefining Thermodynamic Potentials in 2+1 Dimensional Massive Gravity: A Minimal Length Approach

Waheed A. Dar[1] · Nadeem ul Islam[2] · Prince A. Ganai[3]

[1] Department of Physics, National Institute of Technology, Srinagar, Kashmir–190006, India.
[2] Department of Physics, National Institute of Technology, Srinagar, Kashmir–190006, India; Corresponding Author E-mail: drnadeemulislam@gmail.com
[3] Department of Physics, National Institute of Technology, Srinagar, Kashmir–190006, India.

Received: October 30, 2024; **Revised:** December 11, 2024; **Accepted:** December 15, 2024

Abstract. This work delves into the intricate relationship between quantum fluctuations and the thermodynamic properties of a (2+1)-dimensional Anti-de Sitter (AdS) black hole. We adopt the framework of massive gravity to investigate the well-known Bañados-Teitelboim-Zanelli (BTZ) black hole solution. After a concise review of the BTZ solution in this context, we proceed with a detailed derivation of exponentially corrected thermodynamic potentials, with careful consideration of massive gravity effects. Our analysis culminates in a qualitative exploration, highlighting the dependence of these corrected potentials on the event horizon radius through a series of plots. By varying the correction parameter α, which represents the intensity of quantum fluctuations, we uncover diverse behaviors that provide insights into the complex connection between quantum phenomena and black hole thermodynamics.

Keywords: Massive Gravity; Quantum Corrections; Exponential Corrections; Black Hole Thermodynamics; Minimal Length.

Contents

1 Introduction

While General Relativity (GR) reigns supreme in the realm of low-energy phenomena, its shortcomings in addressing dark matter and dark energy pose a significant challenge [1]. This has spurred a scientific quest for alternative frameworks, leading to the captivating theory of massive gravity. In stark contrast to GR's massless graviton, massive gravity postulates a massive spin-2 graviton. This intriguing modification holds immense promise: the momentum-tensor of massive gravitons intriguingly aligns with the properties of dark matter [2–4]. Additionally, massive gravity offers the potential to explain the universe's accelerated expansion without invoking a cosmological constant [5,6]. The concept of massive modes gains further credence from its connection to solutions for the hierarchy problem [7–9]. The theory finds further support in advancements within string theory and approaches to quantum gravity [10–12]. The realm of black hole thermodynamics undergoes a fascinating transformation when viewed through the lens of massive gravity. Delving into these modifications unlocks a treasure trove of knowledge regarding the thermodynamic nature of black holes. This paper embarks on a journey to explore black hole thermodynamics through the captivating lens of massive gravity, specifically focusing on the (2+1)-dimensional BTZ black hole solution. Our primary objective is to elucidate the profound influence of quantum fluctuations on the thermodynamic potentials of this captivating black hole, thereby enriching our comprehension of black hole physics within the framework of massive gravity [13–19].

The profound connection between Hawking's area theorem in General Relativity (GR) and the second law of thermodynamics [20], as brilliantly highlighted by Bekenstein [21–25], paved the way for assigning black holes a maximum entropy. This groundbreaking concept eventually led to the captivating holographic principle [26,27], which posits that the entirety of information within a volume of space resides on its boundary. However, the inescapable presence of quantum fluctuations at the Planck scale throws a wrench into the holographic principle's pristine picture. These fluctuations introduce subtle yet significant corrections to the very fabric of spacetime [28,29]. Consequently, to reconcile the area-entropy relationship with the emergence of the holographic principle, a revision of the Bekenstein entropy formula is necessary to account for quantum gravity's influence. In order to quantitatively evaluate the impact of quantum fluctuations on the Bekenstein-Hawking entropy relation $S = \frac{A}{4}$, a Taylor series expansion is implemented to compute short-distance corrections. This approach allows for a more precise accounting of the entropy beyond the leading-order term proportional to the black hole's event horizon area (A). This method is consistent with the idea that thermal fluctuations in thermodynamics directly translate into quantum fluctuations in spacetime geometry [32], building on the formalism of Jacobson [30,31], who established a connection between thermodynamics and spacetime geometry. Significant efforts have been made to understand quantum fluctuations through both qualitative and quantitative methods. For example, a qualitative analysis in [33] examined the impact of thermal fluctuations on the thermodynamics of the Gödel black hole. In [34], researchers used the Cardy formula to study how quantum gravity corrections influence the thermodynamic states of various black holes.

Additional studies in [35,36] investigated the background matter fields of black holes and consistently found logarithmic corrections, offering deeper insights into these fluctuations. Ref. [37] delved into the effects of quantum fluctuations on dilatonic black holes. Meanwhile, Ref. [38] explored quantum corrections to black hole thermodynamics through a partition function methodology. Investigations into thermal fluctuations encompassed charged AdS black holes [39], BTZ black holes [40–45], and massive black holes within AdS space [46]. Sudhaker et al. made notable contributions by calculating the corrected equations of state for a variety of black holes [47–53]. The generalized uncertainty principle was utilized to derive logarithmic thermodynamic corrections to black holes, which were in agreement with corrections derived via other techniques [54,55]. Faizal et al. employed an adaptive graphene model to examine specific thermodynamic characteristics of black holes, as detailed in Ref. [56]. Subsequent studies focused on the impact of thermal fluctuations on the properties of BTZ black holes in massive gravity [57] and on the thermodynamics of black holes with hyper-scaling violations [58]. The stability of the STU black hole, notably influenced by thermal fluctuations, was discussed in Ref. [59]. Extensive discussions on leading-order thermal fluctuation corrections covered various black holes, such as dumb holes (black hole analogs) [60], singly spinning Kerr-AdS black holes [61], dilatonic black stars [62], and modified Hayward black holes [63]. It is essential to highlight that the quantum fluctuations under examination are fundamentally a consequence of thermal fluctuations [64].

This study investigates the effects of quantum fluctuations on the thermodynamics and stability of $2 + 1$-dimensional BTZ black holes within the framework of massive gravity, providing a comprehensive analysis of their behavior. It reveals that quantum fluctuations dramatically influence the entropy, internal energy, free energy, pressure, and enthalpy of black holes, especially at small horizon radii, while their impact diminishes for larger black holes. Positive correction parameters result in unphysical negative entropy, whereas negative corrections stabilize small black holes, potentially allowing for the formation of remnants after evaporation. The study finds that quantum corrections to internal energy and free energy are substantial for small black holes but become negligible as the horizon radius grows. Similarly, pressure is sensitive to the sign of the correction parameter, with positive corrections causing slight increases and negative corrections inducing decreases, converging to asymptotic values at larger scales. Corrected enthalpy shows asymptotic behavior for small black holes, influenced heavily by quantum fluctuations. A stability analysis using specific heat reveals discontinuities that signal structural or second-order phase transitions; however, the nature of these transitions remains unaffected by quantum corrections. Overall, the study offers new insights into how quantum fluctuations in massive gravity govern the thermodynamic properties and stability of BTZ black holes, particularly emphasizing their significance at small scales and their implications for black hole remnants.

The structure of this paper is outlined as follows:

- **Section II:** Reviews the thermodynamic properties of the (2+1)-dimensional black hole solution in massive gravity, including its mass, temperature, and horizon structure.
- **Section III:** Investigates corrections to the Bekenstein-Hawking entropy arising from thermal fluctuations around equilibrium for the massive BTZ black hole.
- **Section IV:** Analyzes quantum-corrected thermodynamic quantities including the Helmholtz free energy, internal energy, and specific heat capacity of the massive BTZ black hole.
- **Section V:** Studies the thermodynamic stability through quantum-corrected heat capacity and Gibbs free energy.
- **Section VI:** Presents conclusions and discusses implications for black hole thermodynamics in massive gravity theories.

2 BTZ Black Hole Dynamics in Massive Gravity

The BTZ black hole in (2+1) dimensions, when considered within the massive gravity framework, is governed by the following action integral:

$$S = -\frac{1}{16\pi}\int d^3x\sqrt{-g}\left[\mathcal{R} - 2\Lambda + L(\mathcal{F}) + m^2\sum_i c_i\left(U_i(g,f)\right)\right], \tag{2.1}$$

where:

- $L(\mathcal{F})$ is the Lagrangian for the vector gauge field.
- $\mathcal{F} = F_{\mu\nu}F^{\mu\nu}$ is the Maxwell invariant, with $F_{\mu\nu}$ being the Faraday tensor and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the gauge potential.
- Λ stands for the cosmological constant.
- $\mathcal{R}$ represents the scalar curvature.
- m is the mass term.
- f represents the reference metric (fixed symmetric tensor).
- c_i are coupling constants.
- U_i represent symmetric polynomials derived from eigenvalues of $\kappa^\mu_\nu \equiv \sqrt{g^{\mu\alpha}f_{\alpha\nu}}$, which is a 3×3 matrix.

The symmetric polynomials for the first four cases are given by:

$$f_1 = [\kappa], \tag{2.2}$$
$$f_2 = [\kappa]^2 - [\kappa^2], \tag{2.3}$$
$$f_3 = [\kappa]^3 - 3[\kappa][\kappa^2] + 2[\kappa^3], \tag{2.4}$$
$$f_4 = [\kappa]^4 - 6[\kappa^2][\kappa]^2 + 8[\kappa^3][\kappa] + 3[\kappa^2]^2 - 6[\kappa^4], \tag{2.5}$$

where:

- $\sqrt{\kappa}$ denotes the matrix square root satisfying $(\sqrt{A})^\mu_\nu(\sqrt{A})^\nu_\lambda = A^\mu_\lambda$.
- $[\kappa]$ represents the trace operation, where $[\kappa] = \kappa^\mu_\mu$.

The field equations obtained by varying the action are:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \frac{1}{2}g_{\mu\nu}L(F) - 2L_F F_{\mu\rho}F^\rho_\nu + m^2\chi_{\mu\nu} = 0, \qquad \nabla_\mu F^{\mu\nu} = 0, \tag{2.6}$$

where

$$\begin{aligned}\chi_{\mu\nu} = &-\frac{c_1}{2}(f_1 g_{\mu\nu} - \kappa_{\mu\nu})\\ &-\frac{c_2}{2}(f_2 g_{\mu\nu} - 2(f_1\kappa_{\mu\nu} + \kappa^2_{\mu\nu}))\\ &-\frac{c_3}{2}(f_3 g_{\mu\nu} - 3(f_2\kappa_{\mu\nu} + 6f_1\kappa^2_{\mu\nu} - \kappa^3_{\mu\nu}))\\ &-\frac{c_4}{2}(f_4 g_{\mu\nu} - 4(f_3\kappa_{\mu\nu} + 12f_2\kappa^2_{\mu\nu} - 24f_1\kappa^3_{\mu\nu} + 24\kappa^4_{\mu\nu})).\end{aligned} \tag{2.7}$$

We explore the static black hole solution using the following metric ansatz:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2. \tag{2.8}$$

The reference metric is chosen as follows:

$$f_{\mu\nu} = \text{diag}(0, 0, c^2 h_{ij}). \tag{2.9}$$

From this reference metric, we derive:

$$f_1 = n\frac{c_0}{r}, \tag{2.10}$$

$$f_2 = f_3 = f_4 = 0. \tag{2.11}$$

In the three-dimensional scenario, the term contributing to massive gravity originates solely from U_1.

For studying the thermodynamic properties of linearly charged BTZ solutions, we specify: $L(F) = -F$:

$$A_\mu = h(r)\delta^t_\mu. \tag{2.12}$$

From the field equations, we obtain the following differential equation:

$$h'(r) + rh'' = 0, \tag{2.13}$$

where:

- $h'(r)$ denotes the first derivative of h with respect to r.
- $h''(r)$ denotes the second derivative of h with respect to r.

The solution to this differential equation is:

$$h(r) = q \ln\left(\frac{r}{l}\right), \tag{2.14}$$

where:

- q is an integration constant representing the black hole's electric charge.
- l is a characteristic length scale constant.

The electromagnetic field tensor, $F_{tr} = \frac{q}{r}$, exhibits independence from the parameter l. To determine exact expressions for the metric function $f(r)$, equations (2.6) and (2.8) are employed to derive the following coupled differential equations:

$$rf'(r) + 2r^2\Lambda + 2q^2 - m^2cc_1r = 0, \tag{2.15}$$

$$\frac{r^2}{2}f''(r) + \Lambda r^2 - q^2 = 0. \tag{2.16}$$

These equations correspond to the tt (or rr) and $\varphi\varphi$ components, respectively. The solution of this system leads to the metric function:

$$f(r) = -\Lambda r^2 - m_0 - 2q^2 \ln\left(\frac{r}{l}\right) - m^2cc_1r. \tag{2.17}$$

In this formulation, m_0 acts as an integration constant associated with the total mass of the black hole. The derived metric function has been verified to satisfy the field equation (2.6) in all its components.

3 Exponentially Corrected Entropy Dynamics

In the previous section, we derived the metric describing the BTZ black hole in a background of massive gravity. It was established that for the BTZ black hole, only the term corresponding to U_1 contributes significantly, while all other U_i's and their coefficients c_i vanish, i.e., $U_2 = U_3 = U_4 = 0$ and $c_2 = c_3 = c_4 = 0$. The physical metric obtained from our analysis is given by:

$$f(r) = -\Lambda r^2 - m_0 - 2q^2 \ln \frac{r}{l} - m^2 cc_1 r. \tag{3.1}$$

By imposing the condition $f(r)\mid_{r=r_+} = 0$ on the metric function $f(r)$, the mass parameter m_0, directly correlated with the black hole's total mass, can be extracted. Consequently, we obtain:

$$m_0 = -\Lambda r^2 - 2q^2 \ln \frac{r_+}{l} - m^2 cc_1 r_+. \tag{3.2}$$

The event horizons are determined by solving the equation $f(r) = 0$:

$$-\Lambda r^2 - m_0 - 2q^2 \ln \frac{r}{l} - m^2 cc_1 r = 0. \tag{3.3}$$

This equation yields two roots:

- r_+ (larger root): Outer event horizon.
- r_- (smaller root): Inner event horizon.

The primary focus of this investigation is a thermodynamic exploration of the black hole solution. A fundamental thermodynamic parameter, temperature, finds its analogue in black hole mechanics as surface gravity, κ. Hawking's seminal work established a direct correlation between temperature and the metric function through the relation:

$$T_H = \frac{1}{4\pi} \frac{df(r)}{dr}\bigg|_{r=r_+}. \tag{3.4}$$

By applying the metric function from equation (3.1), the Hawking temperature T_H for a BTZ black hole in the context of massive gravity is expressed as:

$$T_H = -\frac{\Lambda r_+}{2\pi} - \frac{q^2}{2\pi r_+} + \frac{m^2 cc_1}{4\pi}. \tag{3.5}$$

According to black hole thermodynamics, the Bekenstein-Hawking entropy S_0 can be determined by:

$$S_0 = \frac{\pi r_+}{2G_3}. \tag{3.6}$$

where G_3 denotes the gravitational constant in three dimensions. For simplicity, we set $G_3 = 1$, leading to:

$$S_0 = \frac{\pi r_+}{2}. \tag{3.7}$$

The explicit dependence of the uncorrected entropy, S_0, on the event horizon radius, r_+, is evident from the given expression. Quantum fluctuations, particularly pronounced in the regime of small event horizon radii as evinced by Taylor series analysis, perturb the system's equilibrium configuration, consequently modifying the precise entropy value. It is well established in the literature [65] that the Bekenstein-Hawking entropy formula necessitates logarithmic corrections to accommodate quantum gravitational effects.

The quantum nature of black holes connects entropy to microstates [66], where changes in microstates cause variations in entropy. For a black hole containing N particles, entropy is determined by counting microstates, expressed in statistical mechanics as:

$$\Omega = \frac{(\Sigma_i n_i)!}{\Pi_i s_i}, \tag{3.8}$$

where each n_i is distributed among s_i configurations such that $\Sigma_i s_i n_i = N$. The expression simplifies to:

$$\Omega = \frac{(\Sigma_i s_i)!}{\Pi_i s_i}. \tag{3.9}$$

The most probable configuration, obtained by varying $\log \Omega$ under the constraint $\delta \Sigma_i s_i n_i = 0$, is given by:

$$s_i = (\Sigma_i s_i) \exp(-\lambda n_i), \tag{3.10}$$

where λ is determined from the constraint $s_i = \Sigma_i \exp(-\lambda n_i) = 1$ for $n_i = 1, 2, 3, \ldots, N$. Perturbative corrections introduce a parameter $\lambda = \ln(2) - 2^N$ for large N, resulting in the entropy:

$$S = \lambda N. \tag{3.11}$$

Neglecting terms of order $\mathcal{O}(2^{-2N})$, the exponentially corrected entropy, when eliminating N using Eq. (27), is expressed as:

$$S = S_0 + e^{-S_0}. \tag{3.12}$$

This equation marks a noteworthy deviation from the conventional Bekenstein-Hawking entropy formula. Here, S_0 denotes the equilibrium entropy, defined by the well-known Bekenstein-Hawking relation:

$$S_0 = \frac{\mathcal{A}}{4\ell_P^2}. \tag{3.13}$$

Substituting Eq. (3.12), the non-perturbatively corrected entropy is:

$$S = \frac{\pi r_+}{2} + \alpha e^{-\frac{1}{2}(\pi r_+)}, \tag{3.14}$$

where α represents a correction parameter. This expression characterizes the corrected entropy of the BTZ black hole. To qualitatively assess the influence of quantum corrections, Figure 1 presents a graphical representation of the corrected entropy as a function of the event horizon radius.

The figure demonstrates that in the limit (i.e., as the correction parameter α approaches zero), the entropy curve converges to the uncorrected form, characterized by a monotonic increase with the event horizon radius. However, incorporating quantum corrections introduces notable deviations, particularly within the regime of small black holes. Negative values of α induce a positive definite entropy, thereby enhancing the stability of the BTZ black hole in this domain. Conversely, positive α values lead to unphysical negative entropy. These findings underscore the predominantly perturbative nature of quantum corrections, exerting a significant impact on the entropy spectrum primarily at the lower end of the event horizon radius spectrum, in accordance with theoretical predictions.

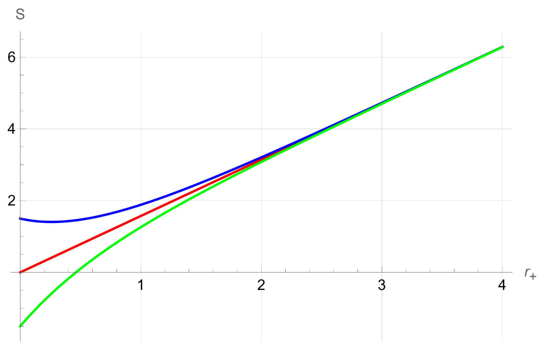

Figure 1: Graph showing the relationship between Entropy and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

4 Exponential Corrections to Classical Thermodynamic Potentials

Internal energy (U) is a fundamental quantity derived from the first law of thermodynamics, which provides insights into the thermodynamic characteristics of a black hole. Typically, the focus is on the change in internal energy rather than its absolute value, as this reveals more about the system's dynamics. To calculate the quantum-corrected internal energy for the BTZ black hole in massive gravity, we leverage the expressions for corrected entropy and Hawking temperature derived earlier. By substituting these quantities into the standard formula for internal energy, we obtain:

$$\begin{aligned} U = \frac{1}{32}\Bigg(& \frac{1}{3} m^2 c c_1 r_+ \left(3\alpha(\pi r_+ - 4) + \pi^2 r_+^2 - 6\pi r_+ + 12\right) \\ & - \frac{1}{6} r_+ \left(8\pi(\alpha - 2)\left(3q^2 + \Lambda r_+^2\right) - 24(\alpha - 1)\Lambda r_+ + 3\pi^2\left(2q^2 r_+ + \Lambda r_+^3\right)\right) \\ & + 8(\alpha - 1) q^2 \ln(r_+)\Bigg), \end{aligned} \tag{4.1}$$

where α signifies the entropy correction parameter addressed earlier. This formula represents the quantum-corrected internal energy of the BTZ black hole in massive gravity, incorporating the effects of the black hole's mass, charge, and cosmological constant, along with an additional quantum correction term that includes $\ln(r)$.

To highlight how quantum corrections modify the internal energy, we graph the relationship between internal energy and the event horizon radius, varying the parameter α to capture the effects of these corrections. As illustrated in Figure 2, when $\alpha = 0$, corresponding to the absence of quantum fluctuations, the internal energy curve matches the uncorrected case and decreases asymptotically as $r \to 0$. For large horizon radii, the internal energy exhibits a monotonous decline. However, with quantum fluctuations considered, the internal energy shows a notable increase at smaller horizon radii. Despite this, at larger radii, the curves for both corrected and uncorrected internal energy converge, suggesting that quantum fluctuations have less impact due to reduced thermal fluctuations in more massive black holes.

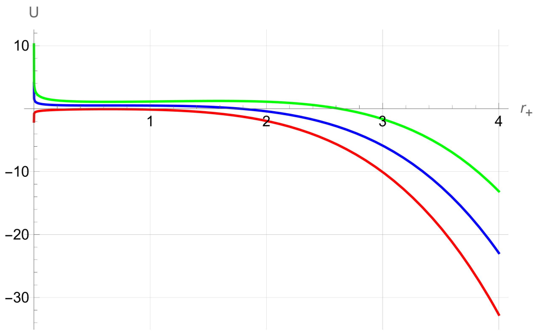

Figure 2: Graph showing the relationship between Internal Energy and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

Following the internal energy calculation, we analyze another key thermodynamic quantity: free energy. This quantity is essential for assessing the stability of a black hole and its potential utility as a heat engine. The free energy is quantitatively expressed by:

$$F = \left\{ \frac{2\alpha q^2}{r_+} + \pi(\alpha - 1)q^2 \log(r_+) - \frac{1}{2}\Lambda r_+(\pi r_+ - 4) + \pi\Lambda r_+^2 \right\}. \tag{4.2}$$

To investigate how quantum fluctuations influence the free energy, we analyze it graphically

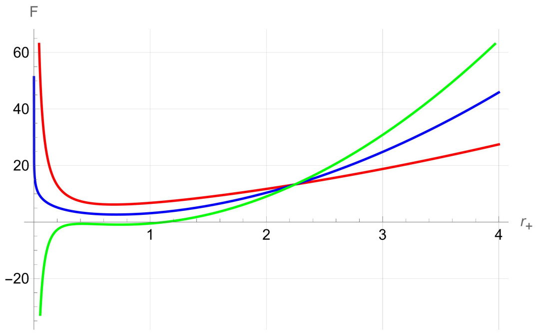

Figure 3: Graph showing the relationship between Free Energy and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

by plotting the free energy expression against the black hole's horizon radius for various values of the correction parameter α. Figure 3 reveals that quantum fluctuations have a significant effect at very small horizon radii. For large black holes, however, the influence of quantum fluctuations diminishes, and the behavior of the corrected and uncorrected curves becomes nearly identical. For smaller black holes, quantum fluctuations lead to a negative asymptotic free energy when α is negative, while positive α values cause the free energy to increase towards positive infinity at these scales.

Having derived the corrected free energy, we can then explore additional thermodynamic quantities such as pressure and enthalpy. In black hole thermodynamics, pressure (P) is not about the force exerted by particles but rather the intensity of the tidal forces experienced by the event horizon. The expression for pressure in the context of the BTZ black hole within massive gravity is given by:

$$P = \left\{ \frac{\alpha(\pi r_+ - 2)\left(\Lambda r_+^2 - q^2\right) + \pi r_+\left(q^2 - 2\Lambda r_+^2\right)}{2\pi r_+^3} \right\}. \tag{4.3}$$

To highlight the distinctions between corrected and uncorrected pressure, we plot the de-

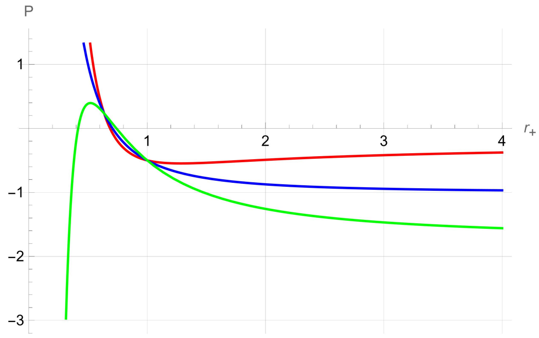

Figure 4: Graph showing the relationship between pressure and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

rived pressure expression (Eq. (4.3)) against the event horizon radius, as illustrated in Figure 4. The graph demonstrates notable variations in pressure behavior as r approaches zero. Specifically, two critical points are identified at small event horizon radii where quantum fluctuations induce subtle but significant changes in pressure. In this regime, positive values of the correction parameter α lead to a slight increase in pressure, whereas negative values result in a decrease. Additionally, prior to the first critical point, positive (negative) values of α correspond to a negative (positive) asymptotic pressure.

With the internal energy and pressure values determined, we can compute the enthalpy using the classical thermodynamic relation $H = U + PV$. Given that internal energy, pressure, and volume are state functions, enthalpy is also a state function. However, our focus extends beyond simply determining the enthalpy of the system; we aim to derive the exponentially corrected enthalpy. With the corrected internal energy and pressure calculated, we can now express the enthalpy as follows:

$$\begin{aligned} H \;=\; & \frac{1}{32}\left\{ \frac{1}{3}\mathrm{cc}_1 m^2 r_+\left(3\alpha(\pi r_+ - 4) + \pi^2 r_+^2 - 6\pi r_+ + 12\right) \right. \\ & + \frac{16\left(\alpha(\pi r_+ - 2)\left(\Lambda r_+^2 - q^2\right) + \pi r_+\left(q^2 - 2\Lambda r_+^2\right)\right)}{r_+} \\ & - \frac{1}{6} r_+\left(8\pi(\alpha - 2)\left(3q^2 + \Lambda r_+^2\right) - 24(\alpha - 1)\Lambda r_+ + 3\pi^2\left(2q^2 r_+ + \Lambda r_+^3\right)\right) \\ & \left. + 8(\alpha - 1)q^2\log(r_+) \right\}. \end{aligned} \tag{4.4}$$

This expression reveals the influence of exponential corrections at small horizon radii. To illustrate this effect, we generate a plot showing the relationship between the corrected enthalpy and the horizon radius for different values of the correction parameter. When quantum corrections are absent (i.e., $\alpha = 0$), the plot reflects the uncorrected enthalpy curve. At larger horizon radii, the behavior of both the uncorrected and corrected enthalpy curves converges, highlighting that quantum fluctuations are negligible at these scales. Conversely, at smaller horizon radii, the effect of quantum fluctuations becomes evident, with the enthalpy exhibiting either positive or negative asymptotic values depending on whether the correction parameter α is positive or negative. To investigate the impact of thermal

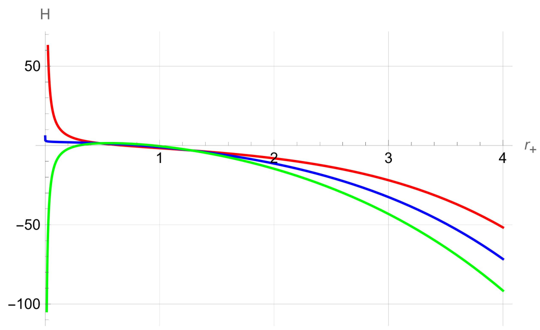

Figure 5: Graph showing the relationship between Enthalpy and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

fluctuations on Gibbs free energy, we begin with its fundamental definition in thermodynamics. The Gibbs free energy represents the maximum work obtainable from a system when temperature and pressure are held constant.

$$G = F + PV, \tag{4.5}$$

with each symbol maintaining its conventional definition. By substituting the corrected values for pressure and free energy obtained from our analysis, we derive the following expression:

$$\begin{aligned} G \;=\; & \frac{1}{32}\Bigg(-\frac{8\left(\alpha e^{-\frac{1}{2}\pi r_+} + \frac{\pi r_+}{2}\right)\left(\mathrm{cc}_1 m^2 - \frac{2q^2}{r_+} - 2\Lambda r_+\right)}{\pi} \\ & + \frac{1}{3}\mathrm{cc}_1 m^2 r_+ \left(3\alpha(\pi r_+ - 4) + \pi^2 r_+^2 - 6\pi r_+ + 12\right) \\ & + \frac{16\left(\alpha(\pi r_+ - 2)\left(\Lambda r_+^2 - q^2\right) + \pi r_+\left(q^2 - 2\Lambda r_+^2\right)\right)}{r_+} \\ & - \frac{1}{6} r_+ \left(8\pi(\alpha - 2)\left(3q^2 + \Lambda r_+^2\right) - 24(\alpha - 1)\Lambda r_+ + 3\pi^2\left(2q^2 r_+ + \Lambda r_+^3\right)\right) \\ & + 8(\alpha - 1) q^2 \log(r_+)\Bigg). \end{aligned} \tag{4.6}$$

This analysis provides a quantitative assessment of how quantum fluctuations affect Gibbs free energy. By comparing the uncorrected and corrected Gibbs free energies, as described by equation 4.6, we can discern notable trends when plotted against the horizon radius. For smaller black holes, the uncorrected Gibbs free energy starts at a higher value and decreases with increasing black hole size. When the correction parameter is positive, the deviation from the uncorrected Gibbs free energy curve is minimal, maintaining a similar decreasing pattern. In contrast, a negative correction parameter results in a negative asymptotic value for Gibbs free energy. Nonetheless, as the black hole size grows, the disparity between uncorrected and corrected Gibbs free energies diminishes.

The expression and plotted data clearly show that as the correction parameter α approaches zero, the uncorrected Gibbs free energy is recovered, consistent with theoretical expectations. This comparative analysis highlights the subtle yet significant impact of quantum fluctuations on the thermodynamic stability and properties of $2+1$ dimensional BTZ black holes in massive gravity.

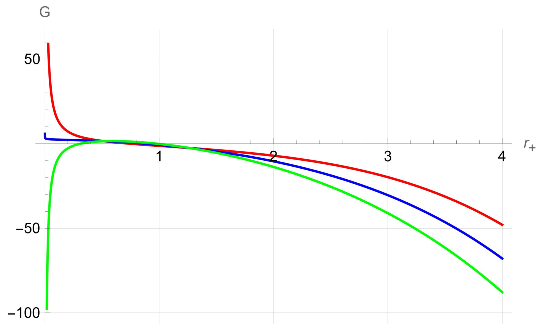

Figure 6: Graph showing the relationship between Gibbs Free Energy and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

5 Phase Transition

The stability of black holes can be assessed by examining their specific heat, which helps determine whether a phase transition occurs within the system. A positive specific heat indicates stability and resistance to phase transitions, while a negative specific heat signals potential instability. By incorporating thermal fluctuations into our analysis, we obtain the expression for specific heat, which defaults to the original, uncorrected form when fluctuations are absent ($\eta = 0$). In classical thermodynamics, specific heat is defined as:

$$C = \frac{dE}{dT}. \tag{5.1}$$

By inserting the corrected values for internal energy (4.2) and Hawking temperature (3.4) into this definition, we can compute the leading-order corrected specific heat for black holes:

$$C = -\frac{\pi r_+ \left(1 - \alpha e^{-\frac{1}{2}\pi r_+}\right)\left(2\left(q^2 + \Lambda r_+^2\right) - \mathrm{cc}_1 m^2 r_+\right)}{2\left(q^2 - 2\Lambda r_+^2\right)}. \tag{5.2}$$

We examine the influence of small statistical fluctuations around equilibrium on system stability by graphing the corrected specific heat, as defined in equation (5.2), versus the event horizon radius.

The data plotted in Fig. 6 indicate that, when quantum fluctuations vanish ($\eta \to 0$), the specific heat exhibits a discontinuity, suggesting the occurrence of a structural or second-order phase transition. Notably, quantum fluctuations do not change the fundamental nature of this transition. This analysis highlights the sensitivity of specific heat to thermal fluctuations and offers important insights into the stability of $2+1$ dimensional BTZ black holes in massive gravity.

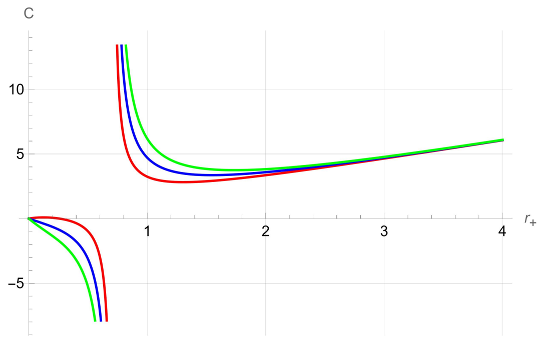

Figure 7: Graph showing the relationship between Specific Heat and the black hole horizon for $q = 1$ and $\Lambda = 1$. The red line represents $\alpha = \frac{3}{2}$, the blue line represents $\alpha = 0$, and the green line represents $\alpha = -\frac{3}{2}$.

6 Final Remarks

In this study, we revisited the foundational concepts of $2+1$ dimensional black hole solutions within the framework of massive gravity. Our investigation focused on quantifying the impact of quantum fluctuations on the thermodynamic properties of BTZ black holes in massive gravity. We began by analyzing the effect of quantum fluctuations on entropy. It was observed that for positive values of the correction parameter, entropy could assume negative values, which are physically non-meaningful. Conversely, negative correction parameters allowed entropy to become positive for small-sized black holes, suggesting that certain quantum fluctuations stabilize evaporated black holes, potentially leaving remnants behind. The comparison of entropy density curves between uncorrected and corrected forms confirmed our assumptions underlying the derivation of corrected entropy. Next, leveraging the derived corrected entropy and Hawking temperature, we computed the corrected internal energy using graphical analysis to discern the impact of quantum fluctuations. Notably, quantum fluctuations prominently influenced the internal energy of small-sized black holes while having negligible effects on larger ones, as evidenced by our findings. Furthermore, corrections to the free energy were evaluated, revealing significant deviations at small event horizon radii corresponding to the behavior observed in entropy due to quantum fluctuations. Similar to internal energy, quantum corrections diminished as the horizon radius increased. Examining the equation of state, we investigated the effect of quantum fluctuations on pressure. Our analysis indicated notable changes in pressure behavior near small event horizon radii, where positive correction parameters led to slight increases while negative corrections induced decreases. These trends were consistent with asymptotic values of pressure affected by the sign of the correction parameter. After examining pressure, internal energy, and free energy, we formulated the corrected enthalpy. This enthalpy, like other thermodynamic properties, demonstrated asymptotic behavior at small horizon radii, influenced by quantum fluctuations. Finally, stability analysis of the BTZ black hole revealed discontinuities in specific heat, indicating the occurrence of structural or second-order phase transitions. Importantly, quantum fluctuations did not alter the nature of these phase transitions, as confirmed through qualitative graphical analysis. In conclusion, this study provides a comprehensive examination of how quantum fluctuations in massive gravity influence the thermodynamic properties and stability of $2+1$ dimensional BTZ black holes, offering insights into their behavior across various scales of horizon radii.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or non-profit sectors.

References

[1] N. Jarosik, et al. "Seven-year wilkinson microwave anisotropy probe (WMAP*) observations: Sky maps, systematic errors, and basic results", The Astrophysical Journal Supplement Series **192**(2), 14 (2011). DOI: 10.1088/0067-0049/192/2/14

[2] K. Aoki and S. Mukohyama, "Massive gravitons as dark matter and gravitational waves", Physical Review D **94**(2), 024001 (2016). DOI: 10.1103/PhysRevD.94.024001

[3] E. Babichev, et al. "Bigravitational origin of dark matter", Physical Review D **94**(8), 084055 (2016). DOI: 10.1103/PhysRevD.94.084055

[4] K. Hinterbichler, "Theoretical aspects of massive gravity", Reviews of Modern Physics **84**(2), 671 (2012). DOI: 10.1103/RevModPhys.84.671

[5] C. Deffayet, "Cosmology on a brane in Minkowski bulk", Physics Letters B **502**(1-4), 199 (2001). DOI: 10.48550/arXiv.hep-th/0010186

[6] C. Deffayet, G. Dvali, and G. Gabadadze, "Accelerated universe from gravity leaking to extra dimensions", Physical Review D **65**(4), 044023 (2002). DOI: 10.1103/PhysRevD.65.044023

[7] G. Dvali, G. Gabadadze, and M. Porrati, "4D gravity on a brane in 5D Minkowski space", Physics Letters B **485**(1-3), 208 (2000). DOI: 10.1016/S0370-2693(00)00669-9

[8] G. Dvali, G. Gabadadze, and M. Porrati, "Metastable gravitons and infinite volume extra dimensions", Physics Letters B **484**(1-2), 112 (2000). DOI: 10.48550/arXiv.hep-th/0002190

[9] G. Dvali and G. Gabadadze, "Gravity on a brane in infinite-volume extra space", Physical Review D **63**(6), 065007 (2001). DOI: 10.1103/PhysRevD.63.065007

[10] H. Gerard'T, "Unitarity in the Brout-Englert-Higgs mechanism for gravity", (2007).[`arXiv:0708.3184 [hep-th]`]

[11] C. P. Burgess, et al. "Warped supersymmetry breaking", Journal of High Energy Physics **2008**(04), 053 (2008). DOI: 10.1088/1126-6708/2008/04/053

[12] N. Vasilis, "Multi-string theories, massive gravity and the AdS/CFT correspondence", Fortschritte der Physik **57**(5-7), 646 (2009). DOI: 10.1002/prop.200900020

[13] A. Bouchareb and G. Clément, "Black hole mass and angular momentum in topologically massive gravity", Classical and Quantum Gravity **24**(22), 5581 (2007). DOI: 10.1088/0264-9381/24/22/018

[14] F. Capela and P. G. Tinyakov, "Black hole thermodynamics and massive gravity", Journal of High Energy Physics **2011**(4), 1 (2011). DOI: 10.1007/JHEP04(2011)042

[15] M. S. Volkov, "Hairy black holes in theories with massive gravitons", Modifications of Einstein's Theory of Gravity at Large Distances. Cham: Springer International Publishing, 161 (2014). DOI: 10.48550/arXiv.1405.1742

[16] B. Eugeny and R. Brito, "Black holes in massive gravity", Classical and Quantum Gravity **32**(15), 154001 (2015). DOI: 10.1088/0264-9381/32/15/154001

[17] S. G. Ghosh, L. Tannukij, and P. Wongjun, "A class of black holes in dRGT massive gravity and their thermodynamical properties", The European Physical Journal C **76**, 1 (2016). DOI: 10.1140/epjc/s10052-016-3943-x

[18] Z. Ming and W.-B. Liu, "Coexistent physics of massive black holes in the phase transitions", (2016). [`arXiv:1610.03648 [gr-qc]`]

[19] H. Ya-Peng, X.-X. Zeng, and H.-Q. Zhang, "Holographic thermalization and generalized Vaidya-AdS solutions in massive gravity", Physics Letters B **765**, 120 (2017). DOI: 10.1016/j.physletb.2016.12.028

[20] S. W. Hawking, "Gravitational radiation from colliding black holes", Physical Review Letters **26**(21), 1344 (1971). DOI: 10.1103/PhysRevLett.26.1344

[21] J. D. Bekenstein, "Black holes and entropy", Physical Review D **7**(8), 2333 (1973). DOI: 10.1103/PhysRevD.7.2333

[22] J. D. Bekenstein, "Generalized second law of thermodynamics in black-hole physics", Physical Review D **9**(12), 3292 (1974). DOI: 10.1103/PhysRevD.9.3292

[23] S. W. Hawking, “Black hole explosions?”, Nature **248**(5443), 30 (1974). DOI: 10.1038/248030a0

[24] S. W. Hawking, “Particle creation by black holes”, Communications in mathematical physics **43**(3), 199 (1975). DOI: 10.1007/BF02345020

[25] N. Altamirano, et al. “Thermodynamics of rotating black holes and black rings: phase transitions and thermodynamic volume”, Galaxies **2**(1), 89 (2014). DOI: 10.3390/galaxies2010089

[26] L. Susskind, “The world as a hologram”, Journal of Mathematical Physics **36**(11), 6377 (1995). DOI: 10.1063/1.531249

[27] R. Bousso, “The holographic principle”, Reviews of Modern Physics **74**(3), 825 (2002). DOI: 10.1103/RevModPhys.74.825

[28] B. Dongsu and S.-J. Rey, “Cosmic holography+”, Classical and Quantum Gravity **17**(15), L83 (2000). DOI: 10.1088/0264-9381/17/15/101

[29] S. K. Rama, “Holographic principle in the closed universe: a resolution with negative pressure matter”, Physics Letters B **457**(4), 268 (1999). DOI: 10.1016/S0370-2693(99)00556-0

[30] T. Jacobson, “Thermodynamics of spacetime: the Einstein equation of state”, Physical Review Letters **75**(7), 1260 (1995). DOI: 10.1103/PhysRevLett.75.1260

[31] R. G. Cai and P. K. Sang “First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe”, Journal of High Energy Physics **2005**(02), 050 (2005). DOI: 10.1088/1126-6708/2005/02/050

[32] B. Pourhassan, F. Mir, and C. Salvatore, “Testing quantum gravity through dumb holes”, Annals of Physics **377**, 108 (2017). DOI: 10.1016/j.aop.2016.11.014

[33] A. Pourdarvish, et al. “Thermodynamics and statistics of Gödel black hole with logarithmic correction”, International Journal of Theoretical Physics **52**, 3560 (2013). DOI: 10.1007/s10773-013-1658-4

[34] T. R. Govindarajan, K. Romesh, and S. Vardarajan, “Quantum gravity on dS3”, Classical and Quantum Gravity **19**(15), 4195 (2002). DOI: 10.1088/0264-9381/19/15/320

[35] R. B. Mann and N. S. Sergey, “Universality of quantum entropy for extreme black holes”, Nuclear Physics B **523**(1-2), 293 (1998). DOI: 10.48550/arXiv.hep-th/9709064

[36] A. J. M. Medved and K. Gabor, “Quantum corrections to the thermodynamics of charged 2D black holes”, Physical Review D **60**(10), 104029 (1999). DOI: 10.1103/PhysRevD.60.104029

[37] J. Jiliang and Y. Mu-Lin, “Statistical entropy of a stationary dilaton black hole from the Cardy formula”, Physical Review D **63**(2), 024003 (2000). DOI: 10.1103/PhysRevD.63.024003

[38] D. Birmingham and S. Siddhartha, “Exact black hole entropy bound in conformal field theory”, Physical Review D **63**(4), 047501 (2001). DOI: 10.1103/PhysRevD.63.047501

[39] B. Pourhassan and F. Mir, “Thermal Fluctuations in a Charged AdS Black Hole”, EPL **111**(4), 40006 (2015). DOI: 10.1209/0295-5075/111/40006

[40] S. Upadhyay, N. Ul Islam, and P. A. Ganai, "A modified thermodynamics of rotating and charged BTZ black hole", Journal of Holography Applications in Physics **2**(1), 25 (2022). DOI: 10.22128/JHAP.2021.454.1004

[41] P. A. Ganai and S. Upadhyay, "Thermal fluctuations to the thermodynamics of a non-rotating BTZ black hole", Progress of Theoretical and Experimental Physics **2019**(10), 103B06 (2019). DOI: 10.1093/ptep/ptz113

[42] N. Ul Islam and P. A. Ganai, "Quantum corrections to AdS black hole in massive gravity", International Journal of Modern Physics A **34**(35), 1950225 (2019). DOI: 10.1142/S0217751X19502257

[43] N. Ul Islam and P. A. Ganai, "First-order corrected thermodynamic potentials characterizing BTZ black hole in massive gravity", International Journal of Modern Physics A **35**(18), 2050080 (2020). DOI: 10.1142/S0217751X20500803

[44] P. A. Ganai and S. Upadhyay, "Thermal fluctuations to the thermodynamics of a non-rotating BTZ black hole", Progress of Theoretical and Experimental Physics **2019**(10), 103B06 (2019). DOI: 10.1093/ptep/ptz113

[45] N. Ul Islam and P. A. Ganai, "Quantum corrections to thermodynamics of BTZ black hole", International Journal of Modern Physics A **34**(11), 1950063 (2019). DOI: 10.1142/S0217751X19500635

[46] B. Pourhassan, H. Farahani, and S. Upadhyay, "Thermodynamics of higher-order entropy corrected Schwarzschild–Beltrami–de Sitter black hole", International Journal of Modern Physics A **34**(28), 1950158 (2019). DOI: 10.1142/S0217751X19501586

[47] S. Upadhyay, "Leading-order corrections to charged rotating AdS black holes thermodynamics", General Relativity and Gravitation **50**(10), 128 (2018). DOI: 10.1007/s10714-018-2459-0

[48] B. Pourhassan, et al. "Exponential corrected thermodynamics of Born–Infeld BTZ black holes in massive gravity", Modern Physics Letters A **37**(33n34), 2250230 (2022). DOI: 10.1142/S0217732322502303

[49] B. Pourhassan, et al. "Quantum gravity effects on Hořava–Lifshitz black hole", Nuclear Physics B **928**, 415 (2018). DOI: 10.1016/j.nuclphysb.2018.01.018

[50] S. Upadhyay, "Quantum corrections to thermodynamics of quasitopological black holes", Physics Letters B **775**, 130 (2017). DOI: 10.1016/j.physletb.2017.10.059

[51] B. Pourhassan, et al. "Thermal fluctuations in a hyperscaling-violation background", The European Physical Journal C **77**, 1 (2017). DOI: 10.1140/epjc/s10052-017-5125-x

[52] S. Upadhyay, S. Soroushfar, and R. Saffari, "Perturbed thermodynamics and thermodynamic geometry of a static black hole in f (R) gravity", Modern Physics Letters A **36**(29), 2150212 (2021). DOI: 10.1142/S0217732321502126

[53] S. Upadhyay and B. Pourhassan, "Logarithmic-corrected van der Waals black holes in higher-dimensional AdS space", Progress of Theoretical and Experimental Physics **2019**(1), 013B03 (2019). DOI: 10.1093/ptep/ptz001

[54] F. Mir and M. M. Khalil, "GUP-corrected thermodynamics for all black objects and the existence of remnants", International Journal of Modern Physics A **30**(22), 1550144 (2015). DOI: 10.1142/S0217751X15501444

[55] A. A. Farag, "No existence of black holes at LHC due to minimal length in quantum gravity", Journal of High Energy Physics **2012**(9), 1 (2012). DOI: 10.1007/JHEP09(2012)067

[56] B. Pourhassan, F. Mir, and S. A. Ketabi, "Logarithmic correction of the BTZ black hole and adaptive model of graphene", International Journal of Modern Physics D **27**(12), 1850118 (2018). DOI: 10.1142/S0218271818300070

[57] B. Pourhassan, et al. "Quantum fluctuations of a BTZ black hole in massive gravity", Physics Letters B **773**, 325 (2017). DOI: 10.1016/j.physletb.2017.08.046

[58] B. Pourhassan, et al. "Thermal fluctuations in a hyperscaling-violation background", The European Physical Journal C **77**, 1 (2017). DOI: 10.1140/epjc/s10052-017-5125-x

[59] B. Pourhassan and F. Mir, "The lower bound violation of shear viscosity to entropy ratio due to logarithmic correction in STU model", The European Physical Journal C **77**, 1 (2017). DOI: 10.1140/epjc/s10052-017-4665-4

[60] B. Pourhassan, F. Mir, and C. Salvatore, "Testing quantum gravity through dumb holes", Annals of Physics **377**, 108 (2017). DOI: 10.1016/j.aop.2016.11.014

[61] B. Pourhassan and F. Mir, "Thermodynamics of a sufficient small singly spinning Kerr-AdS black hole", Nuclear Physics B **913**, 834 (2016). DOI: 10.1016/j.nuclphysb.2016.10.013

[62] B. Pourhassan and F. Mir, "Effect of thermal fluctuations on a charged dilatonic black Saturn", Physics Letters B **755**, 444 (2016). DOI: 10.1016/j.physletb.2016.02.043

[63] B. Pourhassan, F. Mir, and D. Ujjal, "Effects of thermal fluctuations on the thermodynamics of modified Hayward black hole", The European Physical Journal C **76**, 1 (2016). DOI: 10.1140/epjc/s10052-016-3998-8

[64] F. Mir, et al. "Quantum fluctuations from thermal fluctuations in Jacobson formalism", The European Physical Journal C **77**, 1 (2017). DOI: 10.1140/epjc/s10052-016-4575-x

[65] N. Khireddine, "Quantum-corrected black hole thermodynamics to all orders in the Planck length", Physics Letters B **646**(2-3), 63 (2007). DOI: 10.1016/j.physletb.2006.12.072

[66] K. Z. Amin and P. A. Ganai, "Dynamics of a Perturbed Higher Dimensional Black Hole with Exponential Entropy in the Framework of Einstein-Yang-Mills Gravity", International Journal of Theoretical Physics **63**(8), 208 (2024). DOI: 10.1007/s10773-024-05750-4