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Microscopic Analysis of Nuclear Structure and Thermal Behavior in the Super-heavy Region

Mohammad Reza Pahlavani[1] · Asghar Khooy[2]

[1] Department of Nuclear Physics, Faculty of Basic Science, University of Mazandaran, P.O.Box 47415–416, Babolsar, Iran;
Corresponding Author E-mail: m.pahlavani@umz.ac.ir

[2] Department of Nuclear Physics, Faculty of Basic Science, University of Mazandaran, P.O.Box 47415–416, Babolsar, Iran;
E-mail: asghar.khooy@gmail.com

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Abstract. Level density and thermodynamic quantities of $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are calculated based on time dependent pairing energy bach shifted Fermi gas model $TDP-BFGM$. Woods-Saxon potential is considered for the interaction of nucleons inside the nucleus. A temperature dependent pairing energy is also considered. In order to calculate level density and thermodynamic quantities like temperature, entropy and heat capacity of $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes, the level density of these nuclei are calculated by considering the effects of nuclear rotation and vibration. Variation of level density, entropy, temperature and heat capacity as a function of excitation energy for under consideration isotopes are compared by considering the effects of rotation and vibration. Obtained results on variation of heat capacity as a function of excitation energy indicate well the Cooper pair breaking and cooling effects of these super-heavy isotopes. The novelty of this work is the discontinuity in the specific heat at constant volume for these supe-rheavy isotopes that are happen in the excitation energies around 3 MeV for $^{250}Cm_{96}$ and 2.97MeV for $^{260}Fm_{100}$ super-heavy isotopes, which indicates a phase transition from the superfluid state to normal matter.

Keywords: thermodynamic quantities; level density; super-heavy isotopes; temperature; entropy; heat capacity.

1 Introduction

Undoubtedly, one of the old and active titles of nuclear physics is nuclear level density and thermodynamic properties of nuclei. The first theoretical effort to explain the nuclear level density was performed by Bethe in 1936 [1]. In analogy with the classical thermodynamic, he defined thermodynamic quantities of nuclei like temperature, entropy and heat capacity and showed that these quantities are closely related to the nuclear level density. Bethe arranged some experiments to investigate the dependence of nuclear reaction rate on the nuclear level density [2]. Experimentally speaking, the measured data on nuclear level density has been collected in energy region close to nuclear ground levels or Fermi energy [3] and from neutron resonance spacing data [4]. A new method to extract level density and γ-ray strength function from primary γ-ray spectra has been reported by Oslo experimental group [5]. One of the most important applications of nuclear level density is produced in calculations of nuclear reaction cross sections especially in Hauser-Feshbach method [6] which is used in calculation of stellar evolution like supernovae, neutron stars and destruction of compact binary stars [7]. The reaction cross section can also be used to estimate the efficiency of accelerator-driven conversion of nuclear waste. The total radiative strength function can be measured by absorption methods [8]. At energies below the neutron separation energy it can be estimated from radiative neutron capture, usually assuming a model for the nuclear level density. These experiments involve either the total γ-ray spectrum [9] or two-step cascades [10].

Different methods such as the Fermi gas (FGM) model [11], the back-shifted Fermi gas ($BFGM$) model [12,13], the constant temperature (CT) model [14,15], the shell model Monte Carlo approach ($SMMC$) [16,17], the Bardeen Cooper Schrieffer (BCS) model [18, 19],the static path plus random phase approximation (SPA+RPA)algorithm [20,21] and the generalized superfluid model (GSM) [22–25] have been developed to calculate nuclear level density parameter. Among these models, the $BFGM$ approach is commonly used for direct evaluation of the nuclear level density. This method contains two adjustable parameters including the back shift energy, E1, and the single-particle level density parameter a. Using the basic relation of the single-particle level density parameter, as a function of single particle level density at Fermi energy which can be calculated, the $BFGM$ can be parameterized only with one adjustable parameter, $E1$. The single-particle level density can also be obtained through the semiclassical method [26–28] using a proper nuclear mean field potential [29,30].

Thermal properties of nuclei based on experimental and theoretical nuclear level density contain valuable information about phase transitions and cooper pair breaking in nuclei [31–33].

Different applications of nuclear physics, such as nuclear energy generation which depends on fission and fusion reactor design, require accurate knowledge of nuclear reaction rates that are related sensitively to the thermal properties of nuclei. Simulations of astrophysical processes and formation of neutron-reach isotopes that happen in distant galaxies and stars thus require theoretical information about the thermodynamic behavior of nuclear reactions, because some of such reactions did not happen on earth. Efforts to produce highly neutron-rich super-heavy isotopes using rare-isotope accelerator facilities are being happening all over the world. Based on the Fermi gas model of the nucleus,Bethe showed that $\rho(E_x) \propto \exp(2\sqrt{aE_x})$. E_x is the excitation energy. It should be noted that the single-particle level density is evaluated in energies close to the Fermi level. The quantity a is now commonly referred to as the level density parameter. Since then, many reformulations have been done to incorporate realistic single-particle energy spectra for nuclei based on the shell model, like pairing and correlation effects, so that detailed comparisons with the experimental data can be performed.

The atomic nucleus is a complex many-body quantum mechanical system. A complete description of the nuclear structure requires a solution of the Schrödinger equation with a suitable mean field potential in order to obtain nuclear wave functions and nuclear energy levels. Although the strong nuclear interaction is not known well, we still need nuclear models that describe the fundamental properties of the nucleus. The mean field shell model [34] and its extended versions for deformed nuclei and models based on many-body theory [35] are two important groups for extracting single-particle energy levels and wave functions of nucleons inside the nucleus, and studying the collective behaviors of nuclei.

The thermodynamic quantities calculated in this work, particularly the entropy-temperature relationships and phase transitions observed in super-heavy nuclei, may provide valuable benchmarks for testing holographic models of nuclear matter. The AdS/CFT correspondence suggests that strongly coupled quantum systems can be described by classical gravity in higher dimensions. Our results on nuclear level densities and thermal properties could serve as experimental endpoints for validating such holographic descriptions of nuclear structure.

For light nuclei that are small enough, by solving the Schrödinger differential equation with suitable boundary condition, its energy levels can be determined. But for heavy nuclei with high mass number, it is not possible to solve the resultant differential equation. Therefore, statistical microscopic methods combined with spectral explanation and numerical approximations are required to extract the level density.

In order to avoid the complexity of the nuclear level density problem, based on the FG model, some assumptions were used that are not compatible with reality. In other words, effects such as coupling, pairing, shell effect, etc. were not taken into account, actually considering these effects are necessary to accurately determine the nuclear level density. But to account for these interactions the interacting fermions model with BCS potential for pairing effect or the semi-empirical relations of $BSFGM$ is used. The models that deal with the pairing energy, require temperature dependence.

In addition to calculating the reaction cross-section, the nuclear level density can also be used for calculating the thermodynamic quantities of nuclei. In other words, nuclear thermodynamic quantities such as entropy, nuclear temperature and heat capacity can be evaluated using the nuclear level density. Also, the breaking of the first nucleon pair can be seen by calculating the nuclear level density and nuclear heat capacity. The macroscopic GSM [36] constructed based on the behavior of a superfluid at low energy which shows the nuclear phase transition or pair breaking well.

The paper is continued as follows: Theoretical model to calculate the level density and thermodynamic quantities has been presented in Sec. 2. The obtained results have been presented and discussed in Sec. 3. Finally, a brief summary and conclusion are presented in Sec. 4.

2 Theoretical method for calculating thermodynamic properties

The formula of the level density in the $BFGM$ model with an adjustable parameter, E_1 is expressed as follows [28]:

$$\rho(U)_{BFGM} = \frac{1}{12\sqrt{2}\sigma}\frac{\exp(2\sqrt{aU})}{a^{\frac{1}{4}}U^{\frac{5}{4}}}, \tag{1}$$

where the effective excitation energy U by considering shell effect, $E_{shell}(T)$ and temperature dependent pairing energy, $\triangle(T)$ is defined by the following relation

$$U = E(T) - \triangle(T) - E_{shell}(T) - E_1. \quad (2)$$

$E(T)$, a and E_1 are respectively, the excitation energy, the level density parameter and the back shift energy. The spin cut-off factor, σ is defined as follows

$$\sigma^2 = 0.0146 A^{\frac{5}{3}} \frac{1 + \sqrt{1 + 4aU}}{2a}. \quad (3)$$

Temperature dependent shell effects energy can be calculated using the following equation:

$$E_{shell}(E) = M_{Exp} - M_{LDM}, \quad (4)$$

M_{Exp} is the measured value of nuclear mass [38]. M_{LDM} is the nuclear mass in the LDM that is calculated using

$$M_{LDM} = M_n N + M_p P + E_v + E_S + E_C + \triangle(T). \quad (5)$$

Where M_N, M_P, E_v, E_s, E_C, and $\triangle(T)$ are neutron mass, proton mass, volume energy, surface energy, Coulomb energy and pairing energy, respectively. The pairing energy transitions observed here might have corresponding dual descriptions in terms of geometric transitions in the bulk. These energies are defined by the following equation

$$\begin{aligned} E_v &= -C_1 A, \\ E_s &= C_2 A^{\frac{2}{3}}, \\ E_C &= C_3 \frac{Z^2}{A^{\frac{1}{3}}} - C_4 \frac{Z^2}{A}. \end{aligned} \quad (6)$$

With

$$C_i = b_i \left[1 - k \left(\frac{N - Z}{A} \right)^2 \right], \qquad (i = 1, 2). \quad (7)$$

Also, $b_1 = 15.677 MeV$, $b_2 = 18.56 MeV$, $k = 1.79$, $C_3 = 0.717$ and $C_4 = 1.21129$.

The protons and neutrons pairing energy can be calculated using the following equation [39–41].

$$\triangle(0) = \begin{cases} \triangle_n(0) + \triangle_p(0), & \text{for } Z \text{ even and } N \text{ even}, \\ \triangle_p(0), & \text{for } Z \text{ even and } N \text{ odd}, \\ \triangle_n(0), & \text{for } Z \text{ odd and } N \text{ even}, \\ 0, & \text{for } Z \text{ odd and } N \text{ odd}, \end{cases}$$

where $\triangle_n(0)$ and $\triangle_p(0)$ are defined as follows

$$\begin{aligned} \triangle_n(0) &= \frac{r}{N^{\frac{1}{3}}} \exp \left[-s \left(\frac{N - Z}{A} \right) - t \left(\frac{N - Z}{A} \right)^2 \right] \\ \triangle_p(0) &= \frac{r}{Z^{\frac{1}{3}}} \exp \left[s \left(\frac{N - Z}{A} \right) - t \left(\frac{N - Z}{A} \right)^2 \right], \end{aligned} \quad (8)$$

where, $r = 5.72$, $s = 0.118$ and $t = 8.12$ are considered in the calculation. Also, $\triangle(T)$ is related to $\triangle(0)$ by the following relation [42],

$$\triangle(T) = \frac{\triangle(0)}{1 + \exp(\frac{T}{0.03} - \frac{7.37}{0.03\sqrt{A}})}. \quad (9)$$

Then by using the formula of $BSFG$ for $\rho(E)$, one obtains

$$\frac{1}{T}=\frac{dS}{dE}=\frac{d\ln\rho}{dE}=\left(\sqrt{\frac{a}{U}}-\frac{3}{4U}\right)\frac{dU}{dE}, \tag{10}$$

where S and U are entropy and effective excitation energy, respectively. Then by differentiating U as a function of excitation energy E, one obtains

$$\frac{dU}{dE}=1-\frac{d\triangle(T)}{dE}-\frac{dE_{shell}(T)}{dE}=1-\left(\frac{d\triangle(T)}{dT}+\frac{dE_{shell}(T)}{dT}\right)\frac{dT}{dE}. \tag{11}$$

Substituting eq. (11) into eq. (10) yields

$$\frac{1}{T}=\left(\sqrt{\frac{a}{U}}-\frac{3}{4U}\right)\left[1-\left(\frac{d\triangle(T)}{dT}+\frac{dE_{shell}(T)}{dT}\right)\right]\frac{dT}{dE}, \tag{12}$$

In the simple FG model, the excitation energy versus the temperature is defined by

$$E(T)=aT^2. \tag{13}$$

To consider the energy dependence of pairing energy and shell effects, it is customary to consider the excitation energy, $E(T)$ as a polynomial series of the temperature. Here, a complete set of power series up to a third power for excitation energy as a function of temperature is considered,

$$E(T)=a_0+a_1T+a_2T^2+a_3T^3. \tag{14}$$

Then coefficients $a_0,\ldots,a_3$ are obtained by the substitution of $E(T)$ from eq. (14) into eq. (12) in each small interval of temperature. The mass number dependent level density parameter $a(U,A)$is relate with the $BSFGM$ asymptotic level density parameter, $\breve{a}$ as,

$$a(U,A)=\breve{a}\left[1+\frac{1-\exp(-\gamma U)}{U}E_{shell}(T)\right], \tag{15}$$

with

$$\gamma=\frac{0.35}{A^{\frac{1}{3}}}(MeV)^{-1}. \tag{16}$$

The asymptotic level density parameter, $\breve{a}$ is calculated using [28,43,44]

$$\breve{a}=\frac{\pi^2}{6}g, \tag{17}$$

with

$$g=g_n(\epsilon_F^n)+g_n(\epsilon_F^p). \tag{18}$$

Where $g_n(\epsilon_F^n)$ and $g_n(\epsilon_F^p)$ are, respectively, the single-particle level density of neutron and proton in Fermi energy. The level density parameter, a can also be obtained using through fitting with the experimental data or the semi-classical formulas. The calculated level density parameters and thermodynamic quantities could serve as boundary conditions or verification points for holographic models. The single particle level density, $g(\epsilon)$ is evaluated by the following relation,

$$g(\epsilon)=\frac{2}{\pi}\left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}\int drr^2\sqrt{\epsilon-V(r)}\theta(\epsilon-V(r)), \tag{19}$$

where m and ε are the average mass of nucleons (neutron and proton) and single-particle energy levels of nucleons inside the nucleus, respectively. $\theta(X)$ is the well known step function. $V(r)$ is the effective potential that includes nuclear and Coulomb potentials for protons and is equal to nuclear potential for neutrons. Here, Woods-Saxon potential($V_{WS}(r)$) is considered for interaction of nucleons inside the nucleus. The $V_{WS}(r)$ potential for axially symmetric deformed isotope is defined as follows [30,45]

$$V_{WS}(r,\theta) = \frac{V_0}{1+\exp(\frac{r-R(\theta)}{d_s})}. \tag{20}$$

Where R, d_s are nuclear radius and diffuseness parameters, respectively that are defined as follows:

$$\begin{aligned} R(\theta) &= 1.17 + [1+\beta_2 Y_{20}(\theta)]\, R_h, \\ R_h &= (1+0.39I)A^{\frac{1}{3}}, \\ d_s &= 0.50 + 0.33I, \\ I &= \frac{N-Z}{A}. \end{aligned} \tag{21}$$

V_0 is the dept of nuclear Woods-Saxon potential that is obtained using the following relation [46],

$$V_0 = -49.6\left[1 \pm 0.86(\frac{N-Z}{A})\right](MeV), \tag{22}$$

where the $+$ and $-$ signs are used for protons and neutrons, respectively. The Coulomb potential of positively charged protons for axially symmetric isotope is expressed as follows:

$$V_C(r,\theta) = \frac{1}{4\pi\epsilon_0}(\frac{Ze^2}{r})\left[1+(\frac{3R_C^2}{5r^2})\beta_2 Y_{20}(\theta)\right], \tag{23}$$

where Z and β_2 are the atomic number and the quadrupole deformation parameter of isotope, respectively. Following semi-empirical relation [47], is employed to calculate the nuclear charge radius, R_C

$$R_C = r_A\left[1 - b(\frac{N-Z}{A}) + \frac{c}{A}\right]A^{\frac{1}{3}}, \tag{24}$$

with $r_A = 1.235$, $b = 0.177$ and $c = 1.960$. The integral of eq. (19) can not be solved analytically for Woods-Saxon plus Coulomb potential for protons and Woods-Saxon potential for neutrons. Therefore, this integral has been solved numerically to obtain the single-particle level densities of protons and neutrons as a functions of their single-particle energy levels. The effect of Vibrational motion of isotope on the nuclear level density, K_{Vib} is considered by the following relation [48–50]

$$K_{Vib} = \exp(0.0555A^{\frac{2}{3}}T^{\frac{4}{3}}), \tag{25}$$

and the effect of rotational motion of the axially symmetric deformed isotope can be obtained using [13,37]

$$K_{rot} = 0.01389A^{\frac{5}{3}}\sqrt{\frac{U}{a}}(1+\frac{\beta_2}{3}). \tag{26}$$

Finally, by considering the effects of rotational and vibrational motion, the total level density of axially symmetric deformed isotopes is obtained as follows:

$$\rho^{total}(U) = \exp(0.0555A^{\frac{2}{3}}(\frac{u}{a})^{\frac{2}{3}}) \times 0.01389A^{\frac{5}{3}}\sqrt{\frac{U}{a}} \times (1+\frac{\beta_2}{3}) \times \frac{1}{12\sqrt{2}\sigma}\frac{\exp(2\sqrt{aU})}{a^{\frac{1}{4}}U^{\frac{5}{4}}} \tag{27}$$

This formula is used to calculate level density of axially symmetric deformed $^{250}Cm_{96}$ and $^{260}Fm_{100}$ superheavy isotopes based on $BSFGM$ by considering temperature dependent pairing energy and shell effects as well as the effects of rotational and vibrational motions. The entropy of the nucleus can be computed as follows:

$$S = K_B \ln(\frac{\rho}{\rho_0}), \tag{28}$$

where ρ_0 is the normalization constant which has been obtained using the third law of thermodynamics. The nuclear temperature is related to nuclear entropy by the following equation,

$$T = \left(\frac{\partial S}{\partial E}\right)^{-1}, \tag{29}$$

and finally, the heat capacity is calculated using the following formula,

$$C_v = \left(\frac{\partial T}{\partial E}\right)^{-1}. \tag{30}$$

3 Results and discussion

The presented method is applied to calculate asymptotic level density parameter, $\breve{a}$ single particle pairing energy, $\triangle_0$ shell effects energy E_{shell}, quadrupol deformation parameter, β_2 [38,51] and back shift parameter, $E_1(MeV)$ for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes. These parameters are presented in Table 1. As it is clear from this table, there is a considerable difference between the single particle level density parameter of $^{250}Cm_{96}$ even-even super-heavy isotope and $^{260}Fm_{100}$ heavier odd-even super-heavy isotopes. The calculated level densities using parameters of Table 1, by considering effects of vibrational and rotational motion of axially symmetric deformed isotopes as a function of excitation energy for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ isotopes are compared with $TDP - BFGM$ results in figure 1 (a) and (b), respectively. The total nuclear level density by considering the effects of vibrational and rotational motion are compared for two under consideration isotopes in figure 1 (c). This figure indicates that the mode of variation of level density as a function of excitation energy by considering the effects of vibrational and rotational motion is more than the $TDP - BFGM$ results although the level density by considering the effects of vibrational and rotational motion is higher for heavier isotope. Calculated entropy considering the effects of vibrational and rotational motion for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ isotopes as a function of excited energy are compared with the $TDP - BFGM$ results in figure 2 (a) and (b), respectively. The calculated entropy for these isotopes by considering the effects of vibrational and rotational motion are compared in figure 2 (c). As it is obvious from this figure, considering the effects of vibrational and rotational motion caused to increase the nuclear entropy while the method of variation of entropy as a function of excitation energy remains the same. Figure 2 (c) shows that there is a small difference between total entropy of these isotopes in lower energies while the difference increases in higher energies. Also, the entropy of $^{260}Fm_{100}$ isotope is more than $^{250}Cm_{96}$ isotope while the mode of variation is approximation same. In figure 3 (a) and (b) the variation of nuclear temperature by considering the effects of vibrational and rotational motion versus the excitation energy for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are compared with the results of the $TDP - BFGM$, respectively. This figure indicates that unlike to the nuclear level density and entropy, considering the effects of vibrational and rotational motions, cause to decrease the nuclear temperature compared to $TDP - BFGM$. Also, as indicated in figure

3c, in lower energies the temperature of $^{250}Cm_{96}$ is more than $^{260}Fm_{100}$ super-heavy isotope while in higher energies ($E > 12MeV$ the temperature of heavier isotopes goes higher than lighter one. The heat capacity as a function of excited energy has been shown in figure 4 for mentioned above isotopes. This figure obviously indicates the first nucleons pair breaking at 3 MeV and $2.97MeV$ for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes, respectively. One may see from Table 1 that these energies are approximately equal to $2\triangle_0$ for each isotope as expected. Unfortunately, the measurements have not been done yet to use for comparison.

4 Conclusion

This paper is dedicated to the study of level density and thermodynamic properties of $^{250}Cm_{96}$ and $^{260}Fm_{100}$ axially symmetric deformed super-heavy isotopes. In order to obtain level density, it was necessary to calculate single particle parameters of these isotopes. Woods-saxon plus coulomb potentials are considered for interaction of protons while only Woods-Saxon potential is used to obtain single-particle asymptotic level density parameter,$\check{a}$. The single-particle level density parameter is used to calculate level density and thermodynamic quantities based on the semi-classical $TDP - BFGM$ considering time dependent pairing energy and shell effects. Calculated results are presented in four figures. Parts (a) and (b) of these figures indicate that the additive behavior of the nuclear level density and entropy by including the effects of vibrational and rotational motion and decreasing of the nuclear temperature and heat capacity. Also illegal behavior of the nuclear temperature in energies around $3MeV$ for $^{250}Cm_{96}$ and about $E = 2.97MeV$ for $^{260}Fm_{100}$ super-heavy isotopes may reflect the phase transition or first cooper pair breaking. Moreover, figure 4 obviously indicate the first nucleons pair breaking that have been happened at 3 MeV, and 2.97 MeV for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ isotopes, respectively. These results provide important empirical constraints for emerging theoretical frameworks, including holographic approaches to nuclear structure. The clear signatures of phase transitions and thermal behavior we observed in these super-heavy systems could serve as valuable test cases for validating holographic models of nuclear matter, where bulk geometric properties should reproduce the observed thermodynamic behavior through the AdS/CFT correspondence. Due to the lack of experimental data, it was not possible to compare our results with the experimental data but we hope that such results can be a guiding light for future scientists to synthesize new superheavy elements.

Table 1: Single particle and back shifted parameters, $\check{a}$, $\triangle(0)$, E_{shell}, β_2 and $E_1(MeV)$ obtained for axially symmetric $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes.

Isotope	$\check{a}$	$\triangle(0)$	E_{shell}	β_2	$E_1(MeV)$
$^{250}Cm_{96}$	23.93	1.5	0.464	0.250	0.919
$^{260}Fm_{100}$	25.15	1.487	0.617	0.230	0.475

Authors' Contributions

All authors have the same contribution.

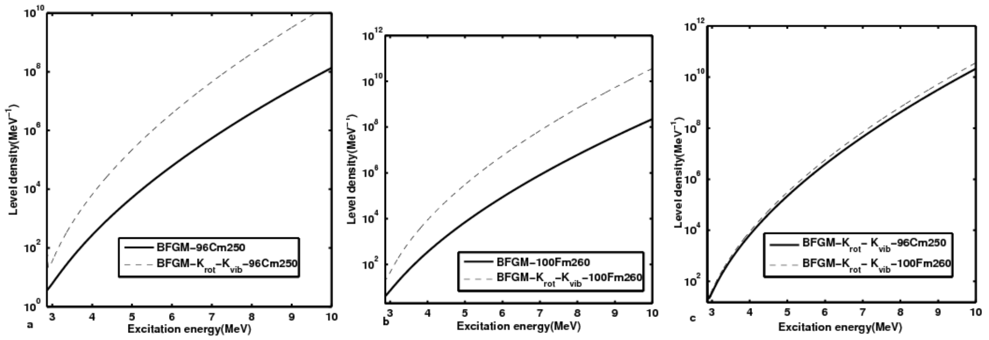

Figure 1: Variation of the calculated nuclear level density considering the effects of rotational and vibrational motion as a function of excited energy compared with the results of $TDP - BFGM$ (a) for $^{250}Cm_{96}$ and (b) for $^{260}Fm_{100}$ super-heavy isotopes. In figure (c) the calculated level density considering the effects of rotational and vibrational motion for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are compared.

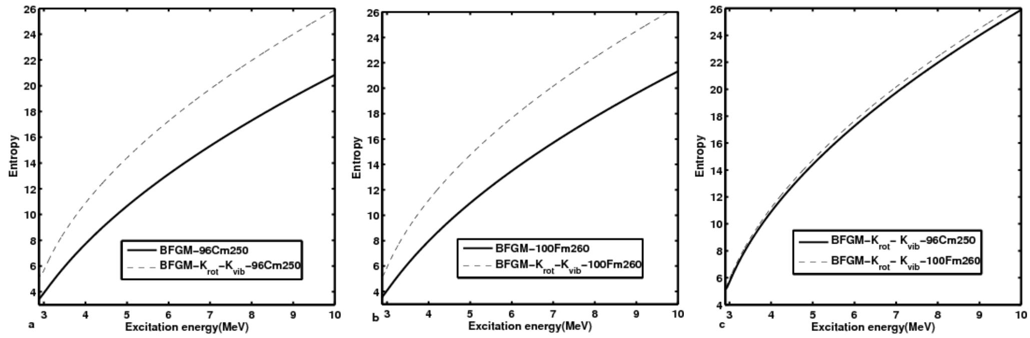

Figure 2: Variation of the calculated nuclear entropy considering the effects of rotational and vibrational motion as a function of excited energy compared with the results of $TDP - BFGM$ (a) for $^{250}Cm_{96}$ and (b) for $^{260}Fm_{100}$ super-heavy isotopes. In (c) the calculated nuclear entropy considering the effects of rotational and vibrational motion for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are compared.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

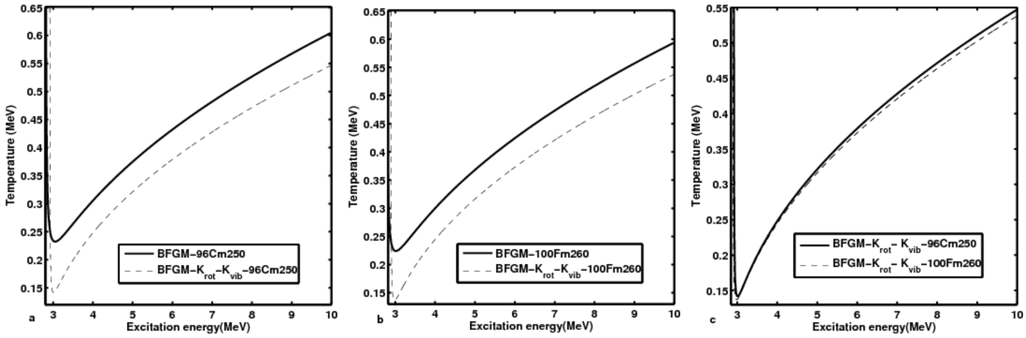

Figure 3: Variation of the calculated nuclear temperature considering the effects of rotational and vibrational motion as a function of excited energy compared with the results of $TDP-BFGM$ (a) for $^{250}Cm_{96}$ and (b) for $^{260}Fm_{100}$ super-heavy isotopes. In (c) the calculated nuclear temperature considering the effects of rotational and vibrational motion for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are compared.

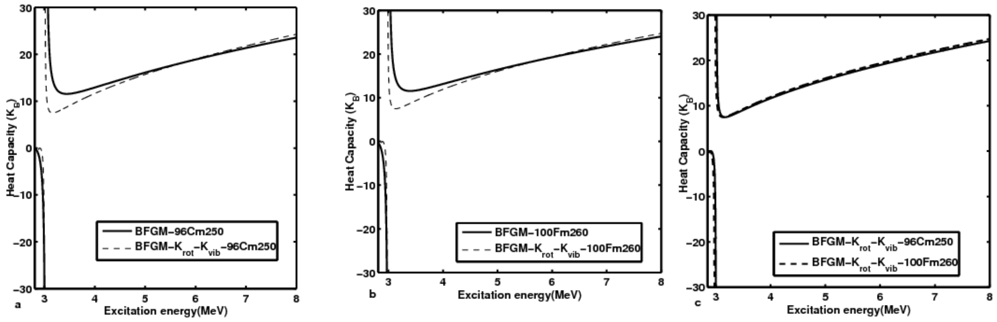

Figure 4: Variation of the calculated nuclear heat capacity considering the effects of rotational and vibrational motion as a function of excited energy compared with the results of $TDP-BFGM$ (a) for $^{250}Cm_{96}$ and (b) for $^{260}Fm_{100}$ super-heavy isotopes. In (c) the calculated nuclear heat capacity considering the effects of rotational and vibrational motion for $^{250}Cm_{96}$ and $^{260}Fm_{100}$ super-heavy isotopes are compared.

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