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Quantum Theory of 3+1 Gravity and Dark Matter: A New Formulation of the Gupta-Feynman based Quantum Field Theory of 3+1 Einstein Gravity

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Abstract. Gravitons and axions play an important role with regard to dark matter. Here, by appeal to the developments of Gupta and Feynman and using a novel mathematical theory based on Ultrahyperfunctions, we are able to provide an exact, quantum relativistic expression for the gravitons and axions self-energies. For a complete explanation of Ultrahyperfunctions and their uses in Quantum Field Theory see the book of Plastino and Rocca.

Ultrahyperfunctions (UHF) are in most cases the generalization and extension to the complex plane of Schwartz tempered distributions. For example, in the book of Plastino and Rocca and in the papers of Bollini et al. you can find a large number of examples of Ultrahyperfunctions. This manuscript is an application to Einstein's Gravity and Dark Matter (EG) of the mathematical theory developed by Bollini et al and continued for more than 25 years by one of the authors of this paper. We will quantize EG using the most general quantization approach, the Schwinger-Feynman variational principle, which is more appropriate and rigorous than the popular functional integral method (FIM). FIM is not applicable here because our Lagrangian contains derivative couplings. We use the Einstein Lagrangian as obtained by Gupta, but we added a new constraint to the theory. Thus the problem of lack of unitarity for the S matrix that appears in the procedures of Gupta and Feynman disappear. Furthermore, we considerably simplify the handling of constraints, eliminating the need to appeal to ghosts for guaranteeing the unitarity of the theory.

Our theory is obviously non-renormalizable. However, this inconvenience is solved by resorting to the theory developed by Bollini et al. This theory is based on the thesis of Alexander Grothendieck and on the theory of Ultrahyperfunctions. Based on these papers, a complete theory has been constructed for 25 years that is able to quantize non-renormalizable Field Theories (FT). Because we are using a Gupta-Feynman based EG Lagrangian and to the new mathematical theory we have avoided the use of ghosts, as we have already mentioned, to obtain a unitary QFT of EG. Moreover the self-energy of the graviton changes its mass and propagator upon interaction with the axion. The mass of the graviton can increase and the bare propagator changes to the dressed propagator. This phenomenon is measurable, but very difficult to detect since the bare mass of the graviton is zero and that of the axion is extremely small. Also, for the first time in the literature, we give explicit formulas for the self-energy of the graviton, interacting and non-interacting with axions. Also, for the first time in the literature, we present 17 graphs corresponding to those self-energies.

Keywords: Quantum Field Theory; Einstein Gravity; Axions; Dark matter; Non-Renormalizable Theories; Unitarity; Ultrahyperfunctions.

1 Introduction

Gravitons and axions are crucial in understanding dark matter. Utilizing advancements by Gupta and Feynman and employing an innovative mathematical framework grounded in Ultrahyperfunctions [1], we derive an exact quantum-relativistic formula for the self-energies of gravitons and axions. For a detailed overview of Ultrahyperfunctions and their role in Quantum Field Theory, refer to [2].

Ultrahyperfunctions (UHF) often represent a generalization and extension of Schwartz's tempered distributions into the complex domain. Numerous examples of UHF applications are presented in our book [2] and in references [3–6]. This work applies the mathematical theory developed by Bollini and collaborators [3–6], which has been refined over more than 25 years, to Einstein's Gravity (EG) and its connection to dark matter. We employ the most comprehensive quantization method, the Schwinger-Feynman variational principle [7], which is more rigorous and suitable than the commonly used functional integral method (FIM). The latter is not applicable here due to the presence of derivative couplings in our Lagrangian.

Our theory builds on the Einstein Lagrangian formulated by Gupta [8–10], with an added constraint that resolves the non-unitary S-matrix issue encountered in Gupta and Feynman's approaches. This adjustment eliminates the need for ghost particles to ensure the theory's unitarity. Although our theory is non-renormalizable, this challenge is addressed by leveraging the framework established by Bollini et al. [3–6], which is rooted in Alexander Grothendieck's Thesis [11] and the theory of Ultrahyperfunctions. Over the past 25 years, this approach has successfully provided a means to quantize non-renormalizable Field Theories (FT). By combining the Gupta-Feynman-based EG Lagrangian with this advanced mathematical approach, we achieve a unitary Quantum Field Theory (QFT) of EG without relying on ghost particles. Furthermore, interactions between gravitons and axions result in modifications to the graviton's mass and propagator. The graviton's mass may increase, and its bare propagator transitions to a dressed propagator. Although these changes are observable, detection is exceedingly challenging due to the graviton's initial mass being zero and the axion's mass being extraordinarily small.

The problem of infinities that appear in a QFT is one of the most important problems that are present in it In particular, in the quantization of gravity, these infinities appear as a consequence of multiplying two distributions at the same point in configuration space and are translated into divergent convolutions in momentum space. These infinities emerge when defining the Lagrangians of the QFT's, since the products of fields that arise in them are products of Vector Distributions (VD), or more generally, Vector Ultrahyperfunctions (VUHF) in the quantum case and products of Ultrahyperfunctions (UHF) in the case of the classic QFt's. This was rigorously established by L. Schwartz in two extensive papers published in the Annales de l'Institut Fourier, [12,13] In them Schwartz makes an extensive and detailed description of the DVs and shows that the product of two of them is not defined, just like the usual distributions. A VD is a continuous linear functional defined on a space of test functions and that takes values in a Locally Convex Topological Vector Space (LCTVS). The appearance of that product, it is what produces the appearance of the infinities in the Lagrangians of the QFT's and these infinities are propagated throughout the resulting theory. In particular in the product of propagators in the phase space, or in its convolution in the momentum space.

More than 25 years ago one of the authors of this manuscript, together with C. G. Bollini, worked to solve this problem using a new mathematical theory: the theory of Ultrahyperfunctions [1]. It was resolved in 4 extensive papers published in IJTP, [3–6] through the development of a new mathematical theory: the Ultrahyperfunctions convolution theory.

The explanation for the use of Ultrahyperfunctions instead of VU is based on the fact that L. Schwartz proved in [12,13] that the products of VD are completely determined if the product of the corresponding distributions over the same test function space is known. To construct this theory Schwartz was based on the theory developed by A. Grothendieck in his thesis [11].

Ultrahyperfunctions are the generalization and extension to the complex plane of the usual distributions defined by L. Schwartz and I. M. Guelfand and are originally known as Ultradistributions of J. Sebastiao e Silva, since they were defined and studied by this extraordinary Portuguese mathematician in an extensive paper published in Mathematische Annalen [1]. Four very interesting papers on Ultrahyperfunctions have been published in [14–18]. Simply put, an Ultrahyperfunction is a pair of analytic functions, one in the upper half-complex plane and one in the lower half-complex plane separated by a strip parallel to the horizontal axis containing the singularities of those functions. Those functions must satisfy certain mathematical conditions that are explained in the references cited above. For example, the Dirac Delta can be written as $\frac{1}{2\pi(x+iy)}$ in the upper half-plane and $\frac{1}{2\pi(x-iy)}$ integrated on the strip counter-clockwise. Furthermore, if we take $y \to 0$ we obtain $\delta(z) = -\frac{1}{2\pi iz}$ which on the real axis is -$\frac{1}{2\pi(x+i0)}$ $+\frac{1}{2\pi(x-i0)}$=$\delta(x)$. Once the convolution of Ultrahyperfunctions is known, the product of distributions is immediately known. Having managed to define a convolution of Ultrahyperfunctions, the infinities of the QFT's do not appear, and thus they are now finite, it is not necessary to regularize the integrals that appear in them, and, furthermore, it is not necessary to renormalize said theories.

To quantize a non-renormalizable QFT is to find an appropriate product of distributions (a product in a ring with zero divisors in the configuration space) an old problem of functional analysis successfully solved in [3–6,19]. At the same time, we keep all existing solutions in the problem of running coupling constants and the renormalization group. With that convolution the UHF space is transformed into a ring with zero-divisors. In it, one has now defined a product between the ring-elements. Thus, any unitary-causal-Lorentz invariant theory quantified in such a manner becomes predictive. The distinction between renormalizable on non-renormalizable QFT's becomes unnecessary now.

In our work we do not use counter-terms to remove infinities from the theory bf because our convolutions are always finite. it Also we don't need to use counter terms, since a non-renormalizable theory involves an infinite number of them. With our convolution, that uses Laurent's expansions (LE) in the parameter employed to define the LE, all finite constants of the convolutions become completely determined, eliminating arbitrary choices of finite constants. The independent term in the Laurent expansion yields the convolution value.

Until now, the attempts to do a QFT of Einstein's Gravity, failed because the quantization of the theory was carried out in: 1) In a Hilbert space with undefined metric; 2) The theory obtained was not unitary; 3) It was not known how to treat non-renormalizable QFTs. The only problem with the Ultrahyperfunctions theory is that it turns out to be extremely complex mathematically. In a first attempt to apply our theory, we achieved a QFT of EG just considering Lorentz Invariant tempered distributions [20] through a simplified version of the UHF convolution [19]. In this manuscript we have managed to make a general QFT of EG, using the theory of UHF to full. Also, the UHF convolution has already been used with success in [21–29].

To achieve this we have resorted to the QFT of EG developed by Suraj N. Gupta [8–10] with a choice of an additional constraint, making a theory similar to that of Quantum Electrodynamics. As a result, we obtain a QFT of EG that is finite and unitary to all perturbative order This was attempted without success first by Gupta and then by Feynman, in his Acta Physica Polonica work [30]. Also, for the first time in the literature, we give explicit formulas for the self-energy of the graviton, interacting and non-interacting with axions. Also, for the first time in the literature, we present 17 graphs corresponding to those self-energies.

In particle physics, the self-energy of a particle is a measure of the energy required to create or destroy the particle. It is a fundamental concept in quantum field theory, which describes the behavior of particles and forces in terms of quantum fields that permeate all of space and time. In quantum field theory, particles are described as excitations or disturbances in their corresponding quantum fields. These fields are constantly fluctuating, and particles can interact with the fluctuations to create or destroy other particles. The self-energy of a particle represents the contribution of these interactions to the total energy of the particle. The self-energy of a particle is related to its mass and charge, and can be calculated using Feynman diagrams, which are graphical representations of the interactions between particles and fields. Feynman diagrams show the different ways in which a particle can interact with its environment, and can be used to calculate the probability of different particle interactions. The self-energy of a particle can have important physical effects, such as changing the mass of the particle and affecting its interactions with other particles. For example, the self-energy of the electron is responsible for the phenomenon of electron screening, which describes how the electrons in a material can shield each other from the electric field of an external charge.

In summary, the self-energy of a particle is a fundamental concept in quantum field theory that describes the energy required to create or destroy the particle, and is related to its interactions with the quantum fields that permeate all of space and time. Note that since we are dealing with a pure four-dimensional theory, we have not considered the possibility of including the dilaton in it.

The manuscript is organized as follows:

- In Section 2, we present Einstein's Lagrangean used in this theory.
- In Section 3, the graviton's self-energy is evaluated up to second order.
- In Section 4, we introduce axiones into our theory and deal with the axions-gravitons interaction.
- In Section 5, we calculate the graviton's self-energy in the presence of axions.
- In Section 6, we evaluate, up to second order, the axion's self-energy.
- Section 7, is dedicated to the conclusions of this work.
- In Appendix A, we present a summary of the definition and some properties of Tempered Ultradistributions.
- In Appendix B, we present a summary of the definition and some properties of Exponential Ultradistributions.
- In Appendix C, we present the preliminary material needed in this paper.
- In Appendix D, we quantize the theory.
- In Appendix E, we discuss the convolution of Ultrahyperfunctions.
- In Appendix F, we obtain a mathematical formula used in this paper.

2 The Gupta-Feynman based Lagrangian of Einstein's QFT

The graviton is a hypothetical elementary particle that is believed to be the carrier of the gravitational force in a quantum mechanical framework. According to the Standard Model of particle physics, which describes the behavior of all known elementary particles and forces, the gravitational force is mediated by a particle called the graviton, which is expected to have zero mass and spin 2.

The idea of the graviton was first proposed in the 1930s by physicists like Paul Dirac and Frits Zernike, who were trying to unify quantum mechanics with general relativity, the theory of gravity developed by Albert Einstein. In general relativity, gravity is described as the curvature of spacetime caused by the presence of massive objects. However, this description is incomplete because it does not take into account the quantum nature of matter and forces.

The graviton is expected to have extremely weak interactions with matter, making it very difficult to detect directly. However, the indirect effects of gravitons can be observed in the behavior of massive objects such as planets, stars, and black holes. Gravitational waves, which were detected for the first time in 2015 by the Laser Interferometer Gravitational-Wave Observatory (LIGO), provide strong evidence for the existence of gravitons.

The search for the graviton is ongoing, and experimental efforts are focused on detecting the tiny fluctuations in spacetime caused by the passage of gravitons. If the graviton is discovered, it would be a major breakthrough in our understanding of the fundamental forces of nature, and would provide important insights into the behavior of black holes, the evolution of the universe, and the nature of spacetime itself. According to Gupta, the Lagrangian of EG is given by [8–10]:

$$\mathcal{L}_G = \frac{1}{\kappa^2}\left[\mathbf{R}\sqrt{|g|} - \frac{1}{2}\eta_{\mu v}\partial_\alpha h^{\mu\alpha}\partial_\beta h^{v\beta}\right], \tag{1}$$

where $\eta^{\mu\nu} = diag(1,1,1,-1)$ and $h^{\mu\nu} = \sqrt{|g|}g^{\mu\nu}$. The effect of the second term in (1) is to fix the gauge. We affect now the linear approximation,

$$h^{\mu v} = \eta^{\mu v} + \kappa\phi^{\mu v}, \tag{2}$$

where κ^2 is the gravitation's constant and $\phi^{\mu v}$ the graviton field. We write then,

$$\mathcal{L}_G = \mathcal{L}_L + \mathcal{L}_I, \tag{3}$$

where

$$\mathcal{L}_L = -\frac{1}{4}\left[\partial_\lambda\phi_{\mu v}\partial^\lambda\phi^{\mu v} - 2\partial_\alpha\phi_{\mu\beta}\partial^\beta\phi^{\mu\alpha} + 2\partial^\alpha\phi_{\mu\alpha}\partial_\beta\phi^{\mu\beta}\right], \tag{4}$$

and, up to 2nd order, one has [8–10],

$$\mathcal{L}_I = -\frac{1}{2}\kappa\phi^{\mu v}\left[\frac{1}{2}\partial_\mu\phi^{\lambda\rho}\partial_v\phi_{\lambda\rho} + \partial_\lambda\phi_{\mu\beta}\partial^\beta\phi_v^\lambda - \partial_\lambda\phi_{\mu\rho}\partial^\lambda\phi_v^\rho\right], \tag{5}$$

where we have made use of the constraint,

$$\phi_\mu^\mu = 0. \tag{6}$$

This constraint is required in order to satisfy gauge invariance [31] As a consequence, the equation of motion of the graviton is given by,

$$\Box\phi_{\mu v} = 0. \tag{7}$$

The solution of the previous equation is given by,

$$\phi_{\mu v} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \left[\frac{a_{\mu v}(\vec{k})}{\sqrt{2k^0}} e^{ik_\mu x^\mu} + \frac{a^+_{\mu v}(\vec{k})}{\sqrt{2k^0}} e^{-ik_\mu x^\mu} \right] d^3k, \tag{8}$$

with $k^0 = |\vec{k}|$.

3 The exact self-energy of the graviton

To evaluate the self-energy (SE) of the graviton, we make use of the generalized Feynman parameters. This is,

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta))}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}}{[Ax+B(1-x)]^{\alpha+\beta}} dx. \tag{9}$$

We now make use of the interaction Hamiltonian $\mathcal{H}_I$. Note that the Lagrangian contains derivative interaction terms,

$$\mathcal{H}_I = \frac{\partial \mathcal{L}_I}{\partial \partial^0 \phi^{\mu\nu}} \partial^0 \phi^{\mu\nu} - \mathcal{L}_I. \tag{10}$$

A typical SE term has the form,

$$\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1}, \tag{11}$$

where $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$, $A = (p-k)^2 - i0$, $\alpha = 1 - \lambda$, $B = \rho - i0$, and $\beta = 1 - \lambda$. As we already said, to evaluate the integral, we use the Feynman parameters. After a Wick rotation, we obtain,

$$\begin{aligned} k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{1-\lambda} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{1-\lambda} = i \int_0^1 x^{-\lambda} (1-x)^{-\lambda} dx \\ \times \frac{\Gamma(2-2\lambda)}{\Gamma^2(1-\lambda)} \int \frac{(p_{\alpha_1} - k_{\alpha_1})(p_{\alpha_2} - k_{\alpha_2}) p_{\alpha_3} p_{\alpha_4}}{[(p-kx)^2 + a]^{2-2\lambda}} d^4p. \end{aligned} \tag{12}$$

Here we have,

$$a = k^2 x(1-x). \tag{13}$$

After the variables-change $u = p - kx$, we find,

$$\begin{aligned} k_{\alpha_1} k_{\alpha_2} (\rho - i0)^{\lambda-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^{\lambda-1} = i \int_0^1 x^{-\lambda} (1-x)^{-\lambda} dx \\ \times \frac{\Gamma(2-2\lambda)}{\Gamma^2(1-\lambda)} \int \frac{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, x, u)}{(u^2+a)^{2-2\lambda}} d^4p, \end{aligned} \tag{14}$$

where f has the form,

$$\begin{aligned} f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, x, u) = & \frac{1}{24} [\eta_{\alpha_1\alpha_2} \eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3} \eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4} \eta_{\alpha_2\alpha_3}] u^4 \\ & + \frac{1}{4} [\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} (1-x)^2 + \eta_{\alpha_1\alpha_3} k_{\alpha_2} k_{\alpha_4} x(x-1) \\ & + \eta_{\alpha_1\alpha_4} k_{\alpha_2} k_{\alpha_3} x(x-1) + \eta_{\alpha_2\alpha_3} k_{\alpha_1} k_{\alpha_4} x(x-1) \end{aligned}$$

$$+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3}x(x-1)+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2}(1-x)^2]u^2$$
$$+k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}x^2(x-1)^2. \quad (15)$$

Evaluating the integral (15) we obtain the following result,

$$\begin{aligned}
k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{\lambda-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{\lambda-1} &= -3i\pi^{\frac{3}{2}}2^{2\lambda}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4}+\eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3})\\
&\frac{\Gamma(3+\lambda)^2\Gamma(1-2\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(1-\lambda)^2\Gamma(6+2\lambda)\Gamma(3+2\lambda)}\Gamma(\lambda)(\rho-i0)^{2+2\lambda}\\
&+i\pi^{\frac{3}{2}}2^{2\lambda}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\\
&\frac{\Gamma(4+\lambda)^2\Gamma(2+\lambda)\Gamma(1-2\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(1-\lambda)^2\Gamma(6+2\lambda)\Gamma(2+2\lambda)}\Gamma(\lambda)(\rho-i0)^{1+2\lambda}\\
&-i\pi^{\frac{3}{2}}2^{2\lambda}(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\\
&\frac{\Gamma(4+\lambda)^2\Gamma(2+\lambda)\Gamma(1-2\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(1-\lambda)^2\Gamma(6+4\lambda)\Gamma(2+2\lambda)}\Gamma(\lambda)(\rho-i0)^{1+2\lambda}\\
&-i\pi^{\frac{3}{2}}2^{2\lambda-1}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\\
&\frac{\Gamma(3+\lambda)^2\Gamma(1-2\lambda)\Gamma\left(\frac{1}{2}+\lambda\right)}{\Gamma(1-\lambda)^2\Gamma(6+2\lambda)\Gamma(1+2\lambda)}\Gamma(\lambda)(\rho-i0)^{2\lambda}. \quad (16)
\end{aligned}$$

3.1 Self-Energy evaluation for $\lambda = 0$

To evaluate SE we must do the Laurent expansion of the preceding result around $\lambda = 0$, according to (75) of Appendix C. We obtain like this,

$$\begin{aligned}
k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{\lambda-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{\lambda-1} &= -i\frac{\pi^2}{5!\lambda}\left\{6(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\rho^2\right.\\
&-[6(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})-4(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}\\
&\left.+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})]\,\rho+2k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\right\}\\
&-\frac{i6\pi^2}{5!}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\left[\ln\rho^2-\frac{137}{30}\right]\rho^2\\
&+i\frac{\pi^2}{5!}\left\{\frac{3}{2}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln\rho^2-\frac{56}{15}\right]\right.\\
&\left.-(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3})\left[\ln\rho^2-\frac{97}{30}\right]\right\}\rho\\
&\left.-i\frac{2\pi^2}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln\rho^2-\frac{47}{30}\right]+\sum_{n=1}^{\infty}a_n\lambda^n\right\}. \quad (17)
\end{aligned}$$

The exact value of the convolution is the independent term of λ in (16). So we get,

$$\begin{aligned}
\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k) &= k_{\alpha_1}k_{\alpha_2}(\rho-i0)^{-1}*k_{\alpha_3}k_{\alpha_4}(\rho-i0)^{-1}\\
&= -\frac{i6\pi^2}{5!}(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4}+\eta_{\alpha_2\alpha_3}\eta_{\alpha_1\alpha_4}+\eta_{\alpha_2\alpha_4}\eta_{\alpha_1\alpha_3})\left[\ln\rho^2-\frac{137}{30}\right]\rho^2\\
&+i\frac{\pi^2}{5!}\left\{\frac{3}{2}(\eta_{\alpha_1\alpha_2}k_{\alpha_3}k_{\alpha_4}+\eta_{\alpha_3\alpha_4}k_{\alpha_1}k_{\alpha_2})\left[\ln\rho^2-\frac{56}{15}\right]\right.
\end{aligned}$$

$$- \left(\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4} + \eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3} + \eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4} + \eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3}\right)\left[\ln\rho^2 - \frac{97}{30}\right]\Bigg\}\rho$$
$$- i\frac{2\pi^2}{5!}k_{\alpha_1}k_{\alpha_2}k_{\alpha_3}k_{\alpha_4}\left[\ln\rho^2 - \frac{47}{30}\right]. \qquad (18)$$

We have to deal with 1296 diagrams of this kind. For simplicity, to make an example of a graph of the self-energy we will consider the momentums equal to k_1.

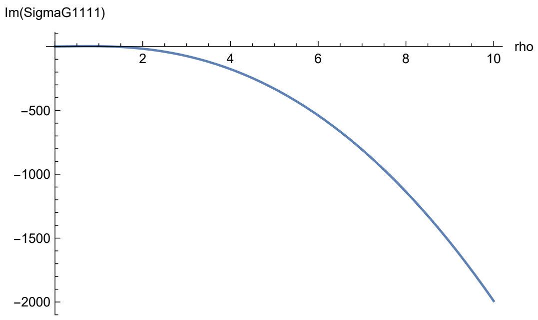

Figure 1: Plot of $\Im[\Sigma_{G1111}]$ versus ρ. (Off-shell mass)

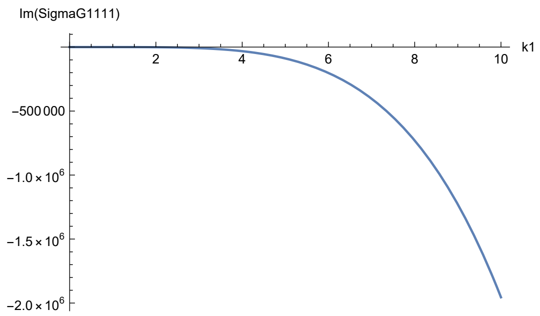

Figure 2: Plot of $\Im[\Sigma_{G1111}]$ versus k_1. (Off-shell mass)

Example of the self-energy with $k_1 = k_3$, and $k_2 = k_4$.

4 Including Axions into the theory

The axion is a hypothetical elementary particle that was proposed in 1977 by Roberto Peccei and Helen Quinn [32] to solve a problem in the Standard Model of particle physics called the strong CP problem. The strong CP problem arises because the strong nuclear force, which binds quarks together to form protons and neutrons, seems to violate a symmetry known as CP symmetry. CP symmetry combines charge conjugation (C) and parity (P) symmetries, which are both fundamental symmetries of nature, and requires that the laws of physics be the same for particles and their antiparticles and for left-handed and right-handed particles.

To explain the strong CP problem, Peccei and Quinn proposed a new symmetry called the Peccei-Quinn symmetry that would require the existence of a new particle, the axion.

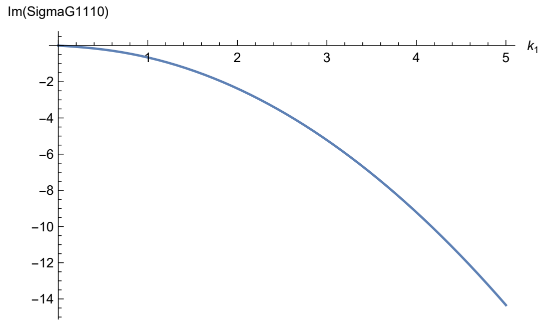

Figure 3: Plot of $\Im[\Sigma_{G1110}]$ versus k_1, with $k_1 = k_2 = k_3$, $\rho = -M_0^2$ and $M_0 = 1$. This is the off-shell mass bradyonic mode.

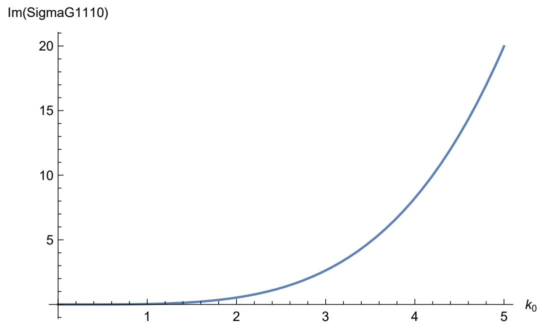

Figure 4: Plot of $\Im[\Sigma_{G1110}]$ versus k_0, with $k_1 = k_2 = k_3$, $\rho = M_0^2$ and $M_0 = 1$. This is the off-shell mass tachyonic mode mode.

The axion would have very low mass (within a certain range) and weak interactions with other particles, making it difficult to detect. However, it would have important implications for astrophysics and cosmology, as it could be a candidate for dark matter, the invisible substance that makes up a significant fraction of the total matter in the universe. As the Dark Matter theory evolved, several experts concluded that the axion could be a candidate for a component of dark matter. It is for this reason that we have included axions in our theory. Thus we have now axions interacting with the graviton. The Lagrangian becomes,

$$\mathcal{L}_{GM} = \frac{1}{\kappa^2}\left[\mathbf{R}\sqrt{|g|} - \frac{1}{2}\eta_{\mu v}\partial_\alpha h^{\mu\alpha}\partial_\beta h^{v\beta}\right] - \frac{1}{2}\left[h^{\mu v}\partial_\mu\phi\partial_v\phi + \sqrt{|g|}m^2\phi^2\right]. \tag{19}$$

The complete Lagrangian now has the form,

$$\mathcal{L}_{GM} = \mathcal{L}_L + \mathcal{L}_I + \mathcal{L}_{LM} + \mathcal{L}_{IM}, \tag{20}$$

where

$$\mathcal{L}_{LM} = -\frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi + m^2\phi^2\right], \tag{21}$$

so that $\mathcal{L}_{IM}$ becomes the interaction Lagrangian for the axion-graviton action,

$$\mathcal{L}_{IM} = -\frac{1}{2}\kappa\phi^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{22}$$

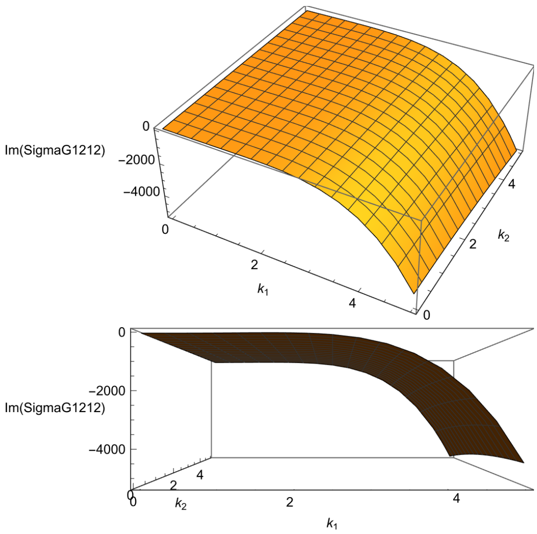

Figure 5: Plot of $\Im[\Sigma_{G1212}]$ versus (k_1, k_2). (Off-shell mass)

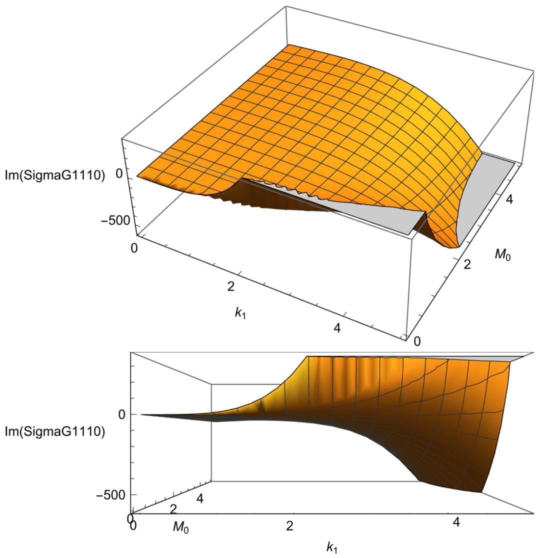

Figure 6: Plot of $\Im[\Sigma_{G1110}]$ versus (k_1, M_0). Off-shell mass bradyonic mode

The new term in the interaction Hamiltonian is,

$$\mathcal{H}_{IM} = \frac{\partial \mathcal{L}_{IM}}{\partial \partial^0 \phi} \partial^0 \phi - \mathcal{L}_{IM}. \tag{23}$$

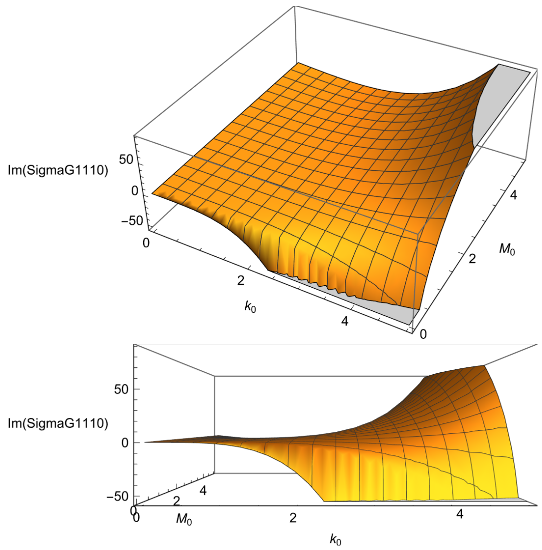

Figure 7: Plot of $\Im[\Sigma_{G1110}]$ versus (k^0, M_0). Off-shell mass tachyonic mode

5 The complete Self-Energy of the Graviton

To evaluate the complete SE, we again resort to generalized Feynman parameters, only in this case the calculation is more complex,

$$\frac{1}{A^\alpha B^\beta C^\gamma D^\delta} = \frac{\Gamma(\alpha+\beta+\gamma+\delta))}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\Gamma(\delta)}\times$$

$$\int_0^1\int_0^1\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}x_1^{\alpha+\beta-1}(1-x_1)^{\gamma-1}x_2^{\alpha+\beta+\gamma-1}(1-x_2)^{\delta-1}}{\{\{[Ax+B(1-x)]x_1+C(1-x_1)\}x_2+D(1-x_2)\}^{\alpha+\beta+\gamma+\delta}}dx\, dx_1\, dx_2, \quad (24)$$

where

$$\begin{aligned} A &= (p-k)^2+m^2-i0, & \alpha &= 1,\\ B &= (p-k)^2-i0, & \beta &= -\lambda,\\ C &= \rho+m^2-i0, & \gamma &= 1,\\ D &= \rho-i0, & \delta &= -\lambda. \end{aligned}$$

The new contribution to the SE of the graviton due to the presence of the axions is given by,

$$\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k) = k_{\alpha_1}k_{\alpha_2}(\rho+m^2-i0)^{-1} * k_{\alpha_3}k_{\alpha_4}(\rho+m^2-i0)^{-1}. \quad (25)$$

After a Wick rotation we obtain,

$$\begin{aligned} &k_{\alpha_1}k_{\alpha_2}(\rho-i0)^\lambda(\rho+m^2-i0)^{-1} * k_{\alpha_3}k_{\alpha_4}(\rho-i0)^\lambda(\rho+m^2-i0)^{-1} =\\ &i\int_0^1\int_0^1\int_0^1 (1-x)^{-\lambda-1}x_1^{-\lambda}x_2^{1-\lambda}(1-x_2)^{-\lambda-1}dx\, dx_1\, dx_2\times\\ &\frac{\Gamma(2-2\lambda)}{\Gamma^2(-\lambda)}\int \frac{p_{\alpha_1}p_{\alpha_2}(k_{\alpha_3}-p_{\alpha_3})(k_{\alpha_4}-p_{\alpha_4})}{[(p-kx_1x_2)^2+a]^{2-2\lambda}}d^4p, \end{aligned} \quad (26)$$

where

$$a = k^2 x_1 x_2 (1 - x_1 x_2) + m^2.(x x_1 x_2 + x_2 - x_1 x_2). \tag{27}$$

After the variables-change $u = p - k x_1 x_2$ we find,

$$\begin{aligned}
&k_{\alpha_1} k_{\alpha_2} (\rho - i0)^\lambda (\rho + m^2 - i0)^{-1} * k_{\alpha_3} k_{\alpha_4} (\rho - i0)^\lambda (\rho + m^2 - i0)^{-1} = \\
&i \int_0^1 \int_0^1 \int_0^1 (1-x)^{-\lambda-1} x_1^{-\lambda} x_2^{1-\lambda} (1-x_2)^{-\lambda-1} dx\, dx_1\, dx_2 \times \\
&\frac{\Gamma(2-2\lambda)}{\Gamma^2(-\lambda)} \int \frac{f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, x_1, x_2, u)}{(u^2 + a)^{2-2\lambda}} d^4 p,
\end{aligned} \tag{28}$$

where f is given by,

$$\begin{aligned}
f(\alpha_1, \alpha_2, \alpha_3, \alpha_4, x_1, x_2, u) =& \frac{1}{24} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] u^4 \\
&+ \frac{1}{4} [\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} (1 - x_1 x_2)^2 + \eta_{\alpha_1\alpha_3} k_{\alpha_2} k_{\alpha_4} x_1 x_2 (x_1 x_2 - 1) \\
&+ \eta_{\alpha_1\alpha_4} k_{\alpha_2} k_{\alpha_3} x_1 x_2 (x_1 x_2 - 1) + \eta_{\alpha_2\alpha_3} k_{\alpha_1} k_{\alpha_4} x_1 x_2 (x_1 x_2 - 1) \\
&+ \eta_{\alpha_2\alpha_4} k_{\alpha_1} k_{\alpha_3} x_1 x_2 (x_1 x_2 - 1) + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2} (1 - x_1 x_2)^2] u^2 \\
&+ k_{\alpha_1} k_{\alpha_2} k_{\alpha_3} k_{\alpha_4} (x_1 x_2)^2 (x_1 x_2 - 1)^2.
\end{aligned} \tag{29}$$

Evaluating the first integral in p and x we obtain for example:

$$\begin{aligned}
&\frac{i\pi^2}{4} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \times \\
&\frac{\Gamma(-2-2\lambda)}{\Gamma(1-\lambda)\Gamma(-\lambda)} \int_0^1 \int_0^{x_1} x_1^{-3-\lambda} y^{3+\lambda} (x-y)^{-1-\lambda} [k^2 x_1 (1-y) + m^2]^{2+2\lambda} \\
&F\left(-2-2\lambda, -\lambda; 1-\lambda; \frac{m^2 x_1}{k^2 x_1 (1-y) + m^2}\right) dx_1 dy.
\end{aligned} \tag{30}$$

Since the integral is convergent at $\lambda = 0$ using our theory, which partly uses Guelfand's regularization [33], we obtain,

$$\frac{i\pi^2}{64} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \times \int_0^1 \int_0^{x_1} x_1^{-3} y^3 (x-y)^{-1} [k^2 x_1 (1-y) + m^2]^2 dx_1 dy. \tag{31}$$

When evaluating this last integral we have,

$$\frac{-i\pi^2}{64} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \left(\frac{5}{2}\rho^2 + 4m^2\rho + \frac{9}{4}m^4\right). \tag{32}$$

The other integrals are calculated in a similar way. The end result is,

$$\begin{aligned}
&k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} * k_{\alpha_3} k_{\alpha_4} (\rho + m^2 - i0)^{-1} = \\
&\frac{-i\pi^2}{64} [\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \eta_{\alpha_1\alpha_3}\eta_{\alpha_2\alpha_4} + \eta_{\alpha_1\alpha_4}\eta_{\alpha_2\alpha_3}] \left(\frac{5}{2}\rho^2 + 4m^2\rho + \frac{9}{4}m^4\right) + \\
&\frac{i\pi^2}{8} [\eta_{\alpha_1\alpha_2} k_{\alpha_3} k_{\alpha_4} + \eta_{\alpha_3\alpha_4} k_{\alpha_1} k_{\alpha_2}] \left(\frac{41}{400}\rho - \frac{3}{2}m^2\right) +
\end{aligned}$$

$$\frac{i\pi^2}{8}[\eta_{\alpha_1\alpha_3}k_{\alpha_2}k_{\alpha_4}+\eta_{\alpha_1\alpha_4}k_{\alpha_2}k_{\alpha_3}+\eta_{\alpha_2\alpha_3}k_{\alpha_1}k_{\alpha_4}+\eta_{\alpha_2\alpha_4}k_{\alpha_1}k_{\alpha_3}]\left(\frac{103}{900}\rho+\frac{35}{144}m^2\right). \quad (33)$$

We have to deal with 1 diagram of this kind.
Accordingly, our desired self-energy total is a combination of $\Sigma_{G\alpha_1\alpha_2\alpha_3\alpha_4}(k)$ and $\Sigma_{GM\alpha_1\alpha_2\alpha_3\alpha_4}(k)$. For simplicity, to make an example of a graph of the self-energy we will consider the momentums equal to k_1, and $m=1$.

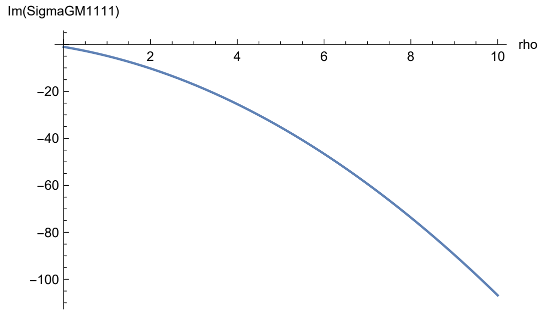

Figure 8: Plot of $\Im[\Sigma_{GM1111}]$ versus ρ. (Off-shell mass)

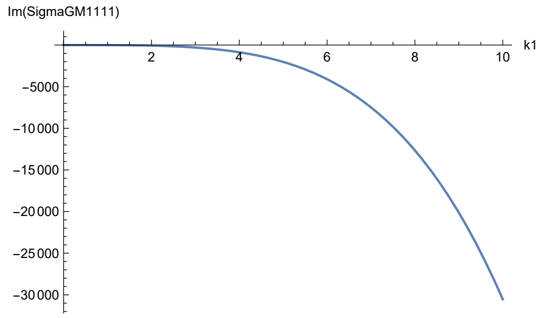

Figure 9: Plot of $\Im[\Sigma_{GM1111}]$ versus k_1. (Off-shell mass)

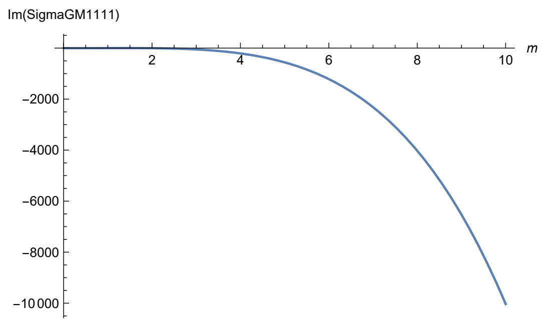

Figure 10: Plot of $\Im[\Sigma_{GM1111}]$ versus m. (Off-shell mass)

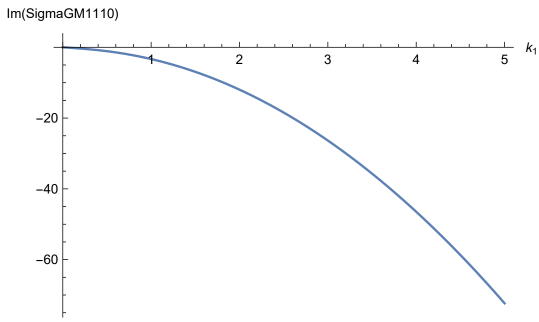

Figure 11: Plot of $\Im[\Sigma_{GM1110}]$ versus k_1, with $k_1 = k_2 = k_3$, $\rho = -M_0^2$, $M_0 = 1$ and $m^2 = 1$ (Off-shell mass bradyonic mode)

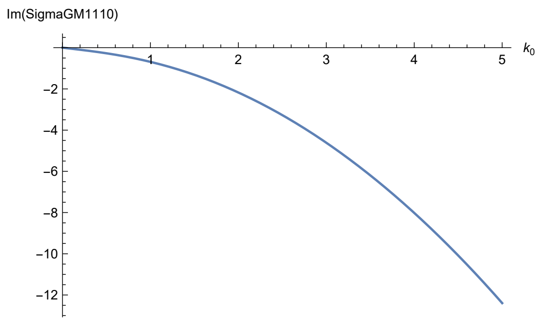

Figure 12: Plot of $\Im[\Sigma_{GM1110}]$ versus k_0, with $k_1 = k_2 = k_3$, $\rho = M_0^2 = 1$ and $m^2 = 1$ (Off-shell mass tachyonic mode)

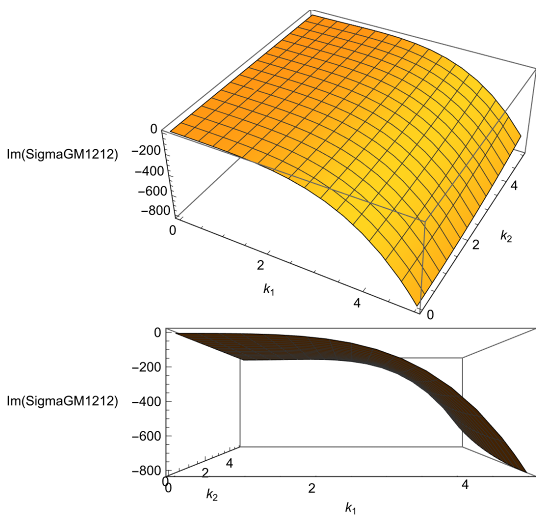

Figure 13: Plot of $\Im[\Sigma_{GM1212}]$ versus (k_1, k_2).

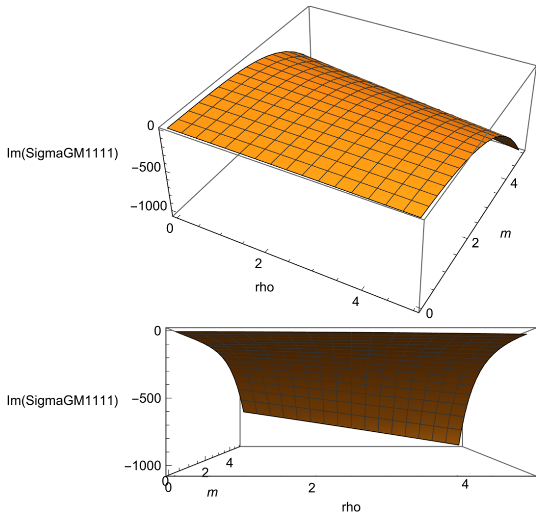

Figure 14: Plot of $\Im[\Sigma_{GM1111}]$ versus (ρ, m).

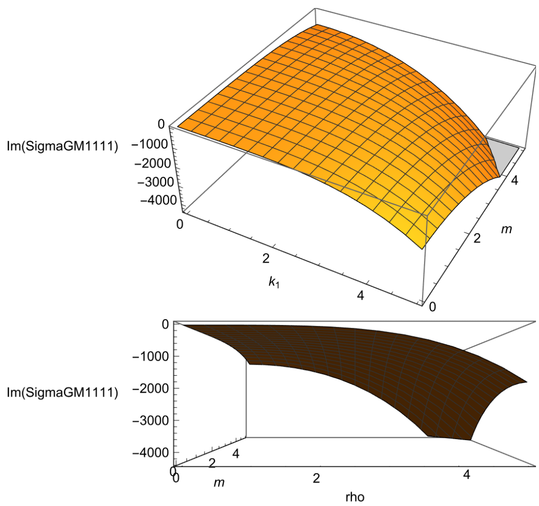

Figure 15: Plot of $\Im[\Sigma_{GM1111}]$ versus (k_1, m).

6 Self Energy of the Axion

We now proceed to evaluate the SE of the axion. A typical term of the self-energy is,

$$\Sigma_{\alpha_1\alpha_2}(k) = k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}. \tag{34}$$

In four dimensions one has,

$$p_{\alpha_1} p_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1}] = \int \frac{p_{\alpha_1} p_{\alpha_2}}{(p^2 + m^2 - i0)[(p-k)^2 - i0]} d^4p. \tag{35}$$

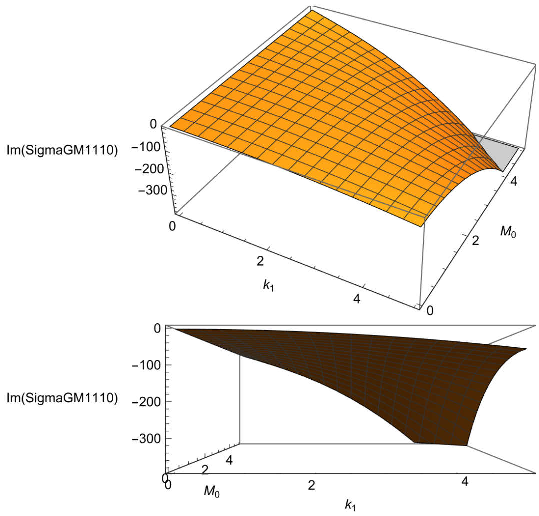

Figure 16: Plot of $\Im[\Sigma_{GM1110}]$ versus (k_1, M_0) with $k_1 = k_2 = k_3$, $\rho = -M_0^2$ and $m = 1$ (Off-shell mass bradyonic mode)

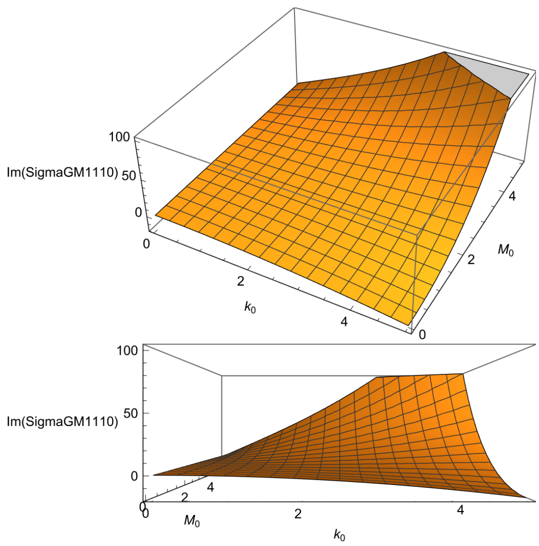

Figure 17: Plot of $\Im[\Sigma_{GM1110}]$ versus (k^0, M_0) with $k_1 = k_2 = k_3$, $\rho = M_0^2$ and $m = 1$ (Off-shell mass tachyonic mode)

With the Feynman generalized parameters used above we obtain,

$$k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} (\rho - i0)^{\lambda} * (\rho - i0)^{\lambda - 1} =$$

$$i \frac{\Gamma(2-2\lambda)}{\Gamma(-\lambda)\Gamma(1-\lambda)} \int_0^1 (1-x)^{-1-\lambda} x_1^{-\lambda} (1-x)^{-\lambda} \int \frac{p_{\alpha_1} p_{\alpha_2}}{[(p-kx_1)^2 + a]^{2-2\lambda}} d^4k dx, \qquad (36)$$

where

$$a = m^2 x(1 - x_1) + k^2 x_1(1 - x_1). \tag{37}$$

We evaluate the integral (36) and find,

$$k_{\alpha_1} k_{\alpha_2} (\rho + m^2 - i0)^{-1} * (\rho - i0)^{-1} = \frac{i\eta_{\alpha_1\alpha_2} \pi^2 m^2}{8}. \tag{38}$$

7 Discussion

In this paper we have performed the Quantum Field Theory of Einstein's gravity and dark matter using a very advanced mathematical theory: the Lorentz Invariant Ultrahyperfunctions convolution theory [3–6]. It is nothing more than having defined a product in a ring with divisors of zero in the configuration space. This theory is not a regularization method. It is a theory apt to quantize non-renormalizable QFT's. More relevant bibliography on the subject are the references [16–18,27–29,34–43].

Since the functional integral is not a suitable mathematical tool to perform the quantization of a theory that contains Ultrahyperfunctions, we have resorted to the more general quantization method for the QFT 's known until now. The variational principle of Feynman and Schwinger. The resulting QFT is finite, unitary, and Lorentz Invariant. As an example of the power of the theory used, we have calculated the SE of the graviton, adding to it the presence of dark matter, represented in this case by axions. It should also be noted that we have added to the QFT of the Gupta-Feynman EG, an additional constraint The addition of this new constraint allows us to make a unitary QFT of the EG. Also, for the first time in the literature, we give explicit formulas for the self-energy of the graviton, interacting and non-interacting with axions. Also, for the first time in the literature, we present 17 graphs corresponding to those self-energies. This theory still needs to be fully explored to see if using it we can obtain new results from the references [44–47].

Authors' Contributions

All authors have contributed equally to the preparation of this manuscript.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

On behalf of all authors, the corresponding author states that there is no conflict of interest in this work.

Ethical Considerations

This work does not contain any studies performed by any other authors.

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1 Appendix A: Tempered Ultradistributions

1.1 Distributions of Exponential Type

For the benefit of the reader, we present here a brief description of the main properties of Tempered Ultradistributions and of Ultradistributions of Exponential Type.
Appendices A and B of this paper were taken from our publication [38] in order to provide an accessible explanation of Ultrahyerfunctions to readers.
Notations. The notations are almost textually taken from Ref. [34]. Let $\mathbb{R}^{\mathbf{n}}$ (respectively $\mathbb{C}^{\mathbf{n}}$) be the real (respectively complex) n-dimensional space whose points are denoted by $x = (x_1, x_2, \dots, x_n)$ (resp. $z = (z_1, z_2, \dots, z_n)$). We shall use the following notations,

(i) $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \alpha x = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

(ii) $x \geqq 0$ means $x_1 \geqq 0, x_2 \geqq 0, \dots, x_n \geqq 0$

(iii) $x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$

(iv) $\mid x \mid = \sum\limits_{j=1}^{n} \mid x_j \mid$

Consider the set of n-tuples of natural numbers $\mathbb{N}^{\mathbf{n}}$. If $p \in \mathbb{N}^{\mathbf{n}}$, then $p = (p_1, p_2, \dots, p_n)$, where p_j is a natural number, $1 \leqq j \leqq n$. $p+q$ denote $(p_1+q_1, p_2+q_2, \dots, p_n+q_n)$ and $p \geqq q$ means $p_1 \geqq q_1, p_2 \geqq q_2, \dots, p_n \geqq q_n$. x^p means $x_1^{p_1} x_2^{p_2} \dots x_n^{p_n}$. We denote by $\mid p \mid = \sum\limits_{j=1}^{n} p_j$ and by D^p we understand the differential operator $\partial^{p_1+p_2+\cdots+p_n}/\partial {x_1}^{p_1} \partial {x_2}^{p_2} \dots \partial {x_n}^{p_n}$.

For any natural number k we define $x^k = x_1^k x_2^k \dots x_n^k$ and $\partial^k/\partial x^k = \partial^{nk}/\partial x_1^k \partial x_2^k \dots \partial x_n^k$. The space $\mathcal{H}$ of test functions such that $e^{p|x|}|D^q\phi(x)|$ is bounded for any natural numbers p and q is defined (Ref. [34]) by means of the countably set of norms:

$$\|\hat{\phi}\|_p = \sup_{0 \leq q \leq p,\, x} e^{p|x|} \left| D^q \hat{\phi}(x) \right|, p = 0, 1, 2, \dots \tag{1}$$

According to reference[48] $\mathcal{H}$ is a $\mathcal{K}\{\mathbf{M_p}\}$ space with:

$$M_p(x) = e^{(p-1)|x|}, p = 1, 2, \dots \tag{2}$$

$\mathcal{K}\{\mathbf{e}^{(\mathbf{p-1})|\mathbf{x}|}\}$ complies condition $(\mathcal{N})$ of Guelfand (Ref. [48]). It is a countable Hilbert and nuclear space:

$$\mathcal{K}\{\mathbf{e}^{(\mathbf{p-1})|\mathbf{x}|}\} = \mathcal{H} = \bigcap_{p=1}^{\infty} \mathcal{H}_{\mathbf{p}}, \tag{3}$$

where $\mathcal{H}_{\mathbf{p}}$ is obtained by completing $\mathcal{H}$ with the norm induced by the scalar product:

$$< \hat{\phi}, \hat{\psi} >_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q \overline{\hat{\phi}}(x) D^q \hat{\psi}(x) \; dx, p = 1, 2, \dots \tag{4}$$

where $dx = dx_1 \; dx_2 \dots dx_n$.
If we take the conventional scalar product:

$$< \hat{\phi}, \hat{\psi} > = \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x) \hat{\psi}(x) \; dx, \tag{5}$$

then $\mathcal{H}$, completed with (5), is the Hilbert space $\mathbf{H}$ of square integrable functions.
By definition, the space of continuous linear functionals defined on $\mathcal{H}$ is the space $\mathbf{\Lambda}_\infty$ of the distributions of the exponential type (Ref. [34]).
The Fourier transform of a distribution of exponential type $\hat{F}$ is given by (see [1,34]):

$$F(k) = \int_{-\infty}^{\infty} H[\Im(k)]H[\Re(x) - H[-\Im(k)]H[-\Re(x)]\hat{F}(x)e^{ikx}\,dx$$
$$= H[\Im(k)]\int_0^{\infty} \hat{F}(x)e^{ikx} - H[-\Im(k)]\int_{-\infty}^{0} \hat{F}(x)e^{ikx}, \tag{6}$$

where F is the corresponding tempered ultradistribution (see the next subsection).
The triplet

$$\mathfrak{H} = (\mathcal{H}, \mathbf{H}, \mathbf{\Lambda}_\infty), \tag{7}$$

is a Rigged Hilbert Space (or a Guelfand's triplet [35]).
Moreover, we have: $\mathcal{H} \subset \mathcal{S} \subset \mathbf{H} \subset \mathcal{S}^{'} \subset \mathbf{\Lambda}_\infty$, where $\mathcal{S}$ is the Schwartz space of rapidly decreasing test functions (Ref. [36]).
Any Rigged Hilbert Space $\mathfrak{G} = (\mathbf{\Phi}, \mathbf{H}, \mathbf{\Phi}^{'})$ has the fundamental property that a linear and symmetric operator on $\mathbf{\Phi}$, which admits an extension to a self-adjoint operator in $\mathbf{H}$, has a complete set of generalized eigenfunctions in $\mathbf{\Phi}^{'}$ with real eigenvalues.

1.2 Tempered Ultradistributions

The Fourier transform of a function $\hat{\phi} \in \mathcal{H}$ is

$$\phi(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\; e^{iz\cdot x}\; dx. \tag{8}$$

Here $\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis.
We call $\mathfrak{H}$ the set of all such functions

$$\mathfrak{H} = \mathcal{F}\left\{\mathcal{H}\right\}. \tag{9}$$

It is a $\mathcal{Z}\{\mathbf{M_p}\}$ countably normed and complete space (Ref. [48]), with:

$$M_p(z) = (1 + |z|)^p, \tag{10}$$

$\mathfrak{H}$ is a nuclear space defined with the norms:

$$\|\phi\|_{pn} = \sup_{z\in V_n} (1 + |z|)^p |\phi(z)|, \tag{11}$$

where $V_k = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^{\mathbf{n}} :|\; Imz_j\; | \leqq k, 1 \leqq j \leqq n\}$.
We can define the habitual scalar product:

$$< \phi(z), \psi(z) >= \int_{-\infty}^{\infty} \phi(z)\psi_1(z)\; dz = \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\; dx, \tag{12}$$

where

$$\psi_1(z) = \int_{-\infty}^{\infty} \hat{\psi}(x)\; e^{-iz\cdot x}\; dx$$

and $dz = dz_1\; dz_2 \ldots dz_n$.

By completing $\mathfrak{H}$ with the norm induced by (12) we obtain the Hilbert space of square integrable functions.
The dual of $\mathfrak{H}$ is the space $\mathcal{U}$ of tempered ultradistributions (Ref. [1,34]). Namely, a tempered ultradistribution is a continuous linear functional defined on the space $\mathfrak{H}$ of entire functions rapidly decreasing on straight lines parallel to the real axis.
The set $\mathfrak{U} = (\mathfrak{H}, \mathbf{H}, \mathcal{U})$ is also a Rigged Hilbert Space. Moreover, we have: $\mathfrak{H} \subset \mathcal{S} \subset \mathbf{H} \subset \mathcal{S}' \subset \mathcal{U}$.
$\mathcal{U}$ can also be characterized in the following way (Ref. [34]): let $\mathcal{A}_\omega$ be the space of all functions $F(z)$ such that:

A) $F(z)$ is analytic on the set $\{z \in \mathbb{C}^{\mathbf{n}} : |Im(z_1)| > p, |Im(z_2)| > p, \dots, |Im(z_n)| > p\}$.

B) $F(z)/z^p$ is bounded continuous in

$$\{z \in \mathbb{C}^{\mathbf{n}} : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, \dots, |Im(z_n)| \geqq p\},$$

where $p = 0, 1, 2, \dots$ depends on $F(z)$.

Let $\mathbf{\Pi}$ be the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^{\mathbf{n}}$. Then $\mathcal{U}$ is the quotient space:

C) $\mathcal{U} = \mathcal{A}_\omega/\mathbf{\Pi}$.

By a pseudo-polynomial we denote a function of z of the form

$$\sum_s z_j^s G(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n),$$

with

$$G(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathcal{A}_\omega.$$

Due to these properties it is possible to represent any ultradistribution as (Ref. [34]):

$$F(\phi) = < F(z), \phi(z) > = \oint_\Gamma F(z)\phi(z)\, dz, \tag{13}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_n$ and where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $F(z)$).
Formula (13) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac Formula" for ultradistributions (Ref. [1]):

$$F(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{f(t)}{(t_1 - z_1)(t_2 - z_2) \dots (t_n - z_n)}\, dt, \tag{14}$$

where the "density" $f(t)$ is the cut of $F(z)$ along the real axis and satisfy:

$$\oint_\Gamma F(z)\phi(z)\, dz = \int_{-\infty}^{\infty} f(t)\phi(t)\, dt. \tag{15}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (15) should be interpreted in the sense of distribution theory.
Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z (Ref. [34]):

$$|F(z)| \leq C|z|^p, \tag{16}$$

where C and p depend on F.
The representation (15) implies that the addition of a pseudo-polynomial $P(z)$ to $F(z)$ do not alter the ultradistribution:

$$\oint_{\Gamma} \{F(z) + P(z)\}\phi(z)\, dz = \oint_{\Gamma} F(z)\phi(z)\, dz + \oint_{\Gamma} P(z)\phi(z)\, dz,$$

But:

$$\oint_{\Gamma} P(z)\phi(z)\, dz = 0,$$

as $P(z)\phi(z)$ is entire analytic in some of the variables z_j (and rapidly decreasing),

$$\therefore \quad \oint_{\Gamma} \{F(z) + P(z)\}\phi(z)\, dz = \oint_{\Gamma} F(z)\phi(z)\, dz. \tag{17}$$

The inverse Fourier transform of (6) is given by:

$$\hat{F}(x) = \frac{1}{2\pi} \oint_{\Gamma} F(k)e^{-ikx}\, dk = \int_{-\infty}^{\infty} f(k)e^{-ikx}\, dx. \tag{18}$$

2 Appendix B: Ultradistributions of Exponential Type

Consider the Schwartz space of rapidly decreasing test functions $\mathcal{S}$. Let Λ_j be the region of the complex plane defined as:

$$\Lambda_j = \{z \in \mathbb{C} : |\Im(z)| < j : j \in \mathbb{N}\}. \tag{19}$$

According to Ref. [1,37] be the space of test functions $\hat{\phi} \in \mathfrak{V}_j$ is constituted by the set of all entire analytic functions of $\mathcal{S}$ for which

$$||\hat{\phi}||_j = \max_{k \le j} \left\{ \sup_{z \in \Lambda_j} \left[e^{(j|\Re(z)|)} |\hat{\phi}^{(k)}(z)| \right] \right\}, \tag{20}$$

is finite.
The space $\mathfrak{Z}$ is then defined as:

$$\mathfrak{Z} = \bigcap_{j=0}^{\infty} \mathfrak{V}_j. \tag{21}$$

It is a complete countably normed space with the topology generated by the set of semi-norms $\{||\cdot||_j\}_{j\in\mathbb{N}}$. The topological dual of $\mathfrak{Z}$, denoted by $\mathfrak{B}$, is by definition the space of ultradistributions of exponential type (Ref. [1,37]). Let $\mathfrak{S}$ be the space of rapidly decreasing sequences. According to Ref. [35] $\mathfrak{S}$ is a nuclear space. We consider now the space of sequences $\mathfrak{P}$ generated by the Taylor development of $\hat{\phi} \in \mathfrak{Z}$

$$\mathfrak{P} = \left\{ \mathfrak{Q} : \mathfrak{Q}\left(\hat{\phi}(0), \hat{\phi}^{'}(0), \frac{\hat{\phi}^{''}(0)}{2}, \ldots, \frac{\hat{\phi}^{(n)}(0)}{n!}, \ldots \right) : \hat{\phi} \in \mathfrak{Z} \right\}. \tag{22}$$

The norms that define the topology of $\mathfrak{P}$ are given by:

$$||\hat{\phi}||_p^{'} = \sup_n \frac{n^p}{n!} |\hat{\phi}^n(0)|, \tag{23}$$

$\mathfrak{P}$ is a subspace of $\mathfrak{S}$ and as consequence is a nuclear space. The norms $||\cdot||_j$ and $||\cdot||_p^{'}$ are equivalent, the correspondence

$$\mathfrak{Z} \Longleftrightarrow \mathfrak{P}, \tag{24}$$

is an isomorphism and therefore $\mathfrak{Z}$ is a countably normed nuclear space. We define now the set of scalar products

$$< \hat{\phi}(z), \hat{\psi}(z) >_n = \sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|z|} \overline{\hat{\phi}^{(q)}}(z) \hat{\psi}^{(q)}(z)\ dz$$
$$= \sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|x|} \overline{\hat{\phi}^{(q)}}(x) \hat{\psi}^{(q)}(x)\ dx. \tag{25}$$

This scalar product induces the norm

$$||\hat{\phi}||_n^{''} = [< \hat{\phi}(x), \hat{\phi}(x) >_n]^{\frac{1}{2}}. \tag{26}$$

The norms $||\cdot||_j$ and $||\cdot||_n^{''}$ are equivalent, and therefore $\mathfrak{Z}$ is a countably hilbertian nuclear space. Thus, if we call now $\mathfrak{Z}_p$ the completion of $\mathfrak{Z}$ by the norm p given in (26), we have:

$$\mathfrak{Z} = \bigcap_{p=0}^{\infty} \mathfrak{Z}_p, \tag{27}$$

where

$$\mathfrak{Z}_0 = \mathbf{H}, \tag{28}$$

is the Hilbert space of square integrable functions.
As a consequence the triplet

$$\mathfrak{U} = (\mathfrak{Z}, \mathbf{H}, \mathfrak{B}), \tag{29}$$

is also a Guelfand's triplet.
$\mathfrak{B}$ can also be characterized in the following way (Refs. [1] and [37]): let $\mathfrak{C}_\omega$ be the space of all functions $\hat{F}(z)$ such that:

A) $\hat{F}(z)$ is an analytic function for $\{z \in \mathbb{C} : |Im(z)| > p\}$.

B)- $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is a bounded continuous function in $\{z \in \mathbb{C} : |Im(z)| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}$ be: $\mathfrak{N} = \{\hat{F}(z) \in \mathfrak{C}_\omega : \hat{F}(z)$ is entire analytic$\}$. Then $\mathfrak{B}$ is the quotient space:

C)- $\mathfrak{B} = \mathfrak{C}_\omega / \mathfrak{N}$

Due to these properties it is possible to represent any ultradistribution of exponential type as (Ref. [34,37]):

$$\hat{F}(\hat{\phi}) =< \hat{F}(z), \hat{\phi}(z) >= \oint_\Gamma \hat{F}(z)\hat{\phi}(z)\ dz, \tag{30}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$). Formula (30) will be our fundamental representation for a ultradistribution of exponential type. The "Dirac Formula" for ultradistributions of exponential type is (Ref. [1,37]):

$$\hat{F}(z) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{t - z}\ dt \equiv \frac{\cosh(\lambda z)}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t - z)\cosh(\lambda t)}\ dt, \tag{31}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint_\Gamma \hat{F}(z)\hat{\phi}(z)\, dz = \int_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t)\, dt. \tag{32}$$

(31) should be used carefully. While $\hat{F}(z)$ is analytic function on Γ, the density $\hat{f}(t)$ is in general singular, so that the right hand side of (32) should be interpreted again in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z (Ref. [34,37]):

$$|\hat{F}(z)| \le C|z|^p e^{p|\Re(z)|} \tag{33}$$

where C and p depend on $\hat{F}$.

The representation (30) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{H}$ to $\hat{F}(z)$ does not alter the ultradistribution:

$$\oint_\Gamma \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\, dz = \oint_\Gamma \hat{F}(z)\hat{\phi}(z)\, dz + \oint_\Gamma \hat{G}(z)\hat{\phi}(z)\, dz.$$

But:

$$\oint_\Gamma \hat{G}(z)\hat{\phi}(z)\, dz = 0,$$

as $\hat{G}(z)\hat{\phi}(z)$ is an entire analytic function,

$$\therefore \quad \oint_\Gamma \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\, dz = \oint_\Gamma \hat{F}(z)\hat{\phi}(z)\, dz. \tag{34}$$

Another very important property of $\mathfrak{B}$ is that $\mathfrak{B}$ is reflexive under the Fourier transform:

$$\mathfrak{B} = \mathcal{F}_c\left\{\mathfrak{B}\right\} = \mathcal{F}\left\{\mathfrak{B}\right\}, \tag{35}$$

where the complex Fourier transform $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by:

$$\begin{aligned} F(k) &= H[\Im(k)] \int_{\Gamma_+} \hat{F}(z)e^{ikz}\, dz - H[-\Im(k)] \int_{\Gamma_-} \hat{F}(z)e^{ikz}\, dz \\ &= \oint_\Gamma \{H[\Im(k)H[\Re(z)] - H[-\Im(k)H[-\Re(z)]\}\hat{F}(z)e^{ikz}\, dz \\ &= H[\Im(k)] \int_0^{\infty} \hat{f}(x)e^{ikx}\, dx - H[-\Im(k)] \int_{-\infty}^{0} \hat{f}(x)e^{ikx}\, dx. \end{aligned} \tag{36}$$

Here Γ_+ is the part of Γ with $\Re(z) \ge 0$ and Γ_- is the part of Γ with $\Re(z) \le 0$. Using (36) we can interpret Dirac's Formula as:

$$F(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{s-k}\, ds \equiv \mathcal{F}_c\left\{\mathcal{F}^{-1}\left\{f(s)\right\}\right\}. \tag{37}$$

The inverse Fourier transform corresponding to (37) is given by:

$$\hat{F}(z) = \frac{1}{2\pi} \oint_\Gamma \{H[\Im(z)]H[-\Re(k)] - H[-\Im(z)]H[\Re(k)]\}F(k)e^{-ikz}\, dk. \tag{38}$$

The treatment for ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to the case of one variable. Thus let Λ_j be given as

$$\Lambda_j = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |\Im(z_k)| \leq j \quad 1 \leq k \leq n\}, \tag{39}$$

and

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{j\left[\sum\limits_{p=1}^{n} |\Re(z_p)|\right]} \left| D^{(k)} \hat{\phi}(z) \right| \right] \right\}, \tag{40}$$

where $D^{(k)} = \partial^{(k_1)} \partial^{(k_2)} \cdots \partial^{(k_n)}$, $k = k_1 + k_2 + \cdots + k_n$.
$\mathfrak{B}^n$ is characterized as follows. Let $\mathfrak{C}^n_\omega$ be the space of all functions $\hat{F}(z)$ such that:

A$'$) $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C}^{\mathbf{n}} : |Im(z_1)| > p, |Im(z_2)| > p, \ldots, |Im(z_n)| > p\}$.

B$'$) $\hat{F}(z) e^{-\left[p \sum\limits_{j=1}^{n} |\Re(z_j)|\right]} / z^p$ is bounded continuous in

$$\{z \in \mathbb{C}^{\mathbf{n}} : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, \ldots, |Im(z_n)| \geqq p\},$$

where $p = 0, 1, 2, \ldots$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}^n$ be: $\mathfrak{N}^n = \{\hat{F}(z) \in \mathfrak{C}^n_\omega : \hat{F}(z)$ is entire analytic function at minus in one of the variables $z_j, 1 \leq j \leq n\}$. Then $\mathfrak{B}^n$ is the quotient space:

C$'$) $\mathfrak{B}^n = \mathfrak{C}^n_\omega / \mathfrak{N}^n$.

We have now

$$\hat{F}(\hat{\phi}) = < \hat{F}(z), \hat{\phi}(z) > = \oint_\Gamma \hat{F}(z) \hat{\phi}(z) \, dz, \tag{41}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$ and where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Again the path Γ surrounds all the singularities of $\hat{F}(z)$). The n-dimensional Dirac's Formula is now

$$\hat{F}(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2) \ldots (t_n - z_n)} \, dt, \tag{42}$$

and the "density" $\hat{f}(t)$ is such that

$$\oint_\Gamma \hat{F}(z) \hat{\phi}(z) \, dz = \int_{-\infty}^{\infty} \hat{f}(t) \hat{\phi}(t) \, dt. \tag{43}$$

The modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C |z|^p e^{\left[p \sum\limits_{j=1}^{n} |\Re(z_j)|\right]}, \tag{44}$$

where C and p depend on $\hat{F}$.

3 Appendix C: Preliminary Materials Needed

In this paper we will not use the functional integral to quantify the gravitational field for two reasons: 1) It does not serve to treat Ultrahyperfunctions, since it cannot take into account the singularities that said Ultrahyperfunctions have in a strip that surrounds the real axis. 2) The interacting Lagrangean has derivative couplings of the graviton field. Instead, we will use the most general method of quantization known, which is the Variational Schwinger-Feynman Method [7] which is able to deal even with high order supersymmetric theories, as exemplified by [49,50]. Such theories can not be quantized with the usual Dirac-brackets technique.
For that purpose, we write the action for a set of fields in the form:

$$\mathcal{S}[\sigma(x), \sigma_0, \phi_A(x)] = \int_{\sigma_0}^{\sigma(x)} \mathcal{L}[\phi_A(\xi), \partial_\mu \phi_A(\xi), \xi] d\xi, \tag{45}$$

where $\sigma(x)$ if a space-like surface passing through the point x. σ_0 is a surface at the remote past, at which all field variations vanish. The Schwinger-Feynman variational principle establishes that:
"Any Hermitian infinitesimal variation $\delta\mathcal{S}$ of the action induces a canonical transformation of the vector space in which the quantum system is defined, and the generator of this transformation is this same operator $\delta\mathcal{S}$".
As a consequence of this statement we obtain [7]:

$$\delta\phi_A = i[\delta\mathcal{S}, \phi_A]. \tag{46}$$

Thus, for a Poincare transformation we have

$$\delta\mathcal{S} = a^\mu \mathcal{P}_\mu + \frac{1}{2} a^{\mu v} \mathcal{M}_{\mu v}. \tag{47}$$

Therefore, the variation of the field is given by:

$$\delta\phi_a = a^\mu \hat{P}_\mu \phi_A + \frac{1}{2} a^{\mu v} \hat{M}_{\mu v} \phi_A. \tag{48}$$

From (46), (47) and (48) we obtain

$$\partial_\mu \phi_A = i[\mathcal{P}_\mu, \phi_A]. \tag{49}$$

In particular $\mu = 0$ we have:

$$\partial_0 \phi_A = i[\mathcal{P}_0, \phi_A]. \tag{50}$$

This result is used to quantize the QFT's. In particular we will use it to quantize the EG.

4 Appendix D: The correct quantization of the theory

We need remember some usual definitions. The energy-momentum tensor is given by

$$T_\rho^\lambda = \frac{\partial\mathcal{L}}{\partial\partial^\rho\phi^{\mu v}} \partial^\lambda \phi^{\mu v} - \delta_\rho^\lambda \mathcal{L}. \tag{51}$$

From this definition we obtain the time-component of the four-momentum vector

$$\mathcal{P}_0 = \int T_0^0 \, d^3x. \tag{52}$$

Using the expression (4) of the Lagrangian of the free fields we obtain:

$$T_0^0 = \frac{1}{4}[\partial_0\phi_{\mu v}\partial^0\phi^{\mu v} + \partial_j\phi_{\mu v}\partial^j\phi^{\mu v} - 2\partial_\alpha\phi_{\mu 0}\partial^0\phi^{\mu\alpha} - 2\partial_\alpha\phi_{\mu j}\partial^j\phi^{\mu\alpha} + 2\partial_\alpha\phi^{\mu\alpha}\partial_0\phi_\mu^0 + 2\partial_\alpha\phi^{\mu\alpha}\partial_j\phi_\mu^j]. \tag{53}$$

Consequently, from this last equation we arrive at:

$$\mathcal{P}_0 = \frac{1}{4}\int k_0 \left[a_{\mu v}(\vec{k})a^{+\mu v}(\vec{k}) + a^{+\mu v}(\vec{k})a_{\mu v}(\vec{k})\right] d^3k. \tag{54}$$

We now use the equation (50) and we have

$$[\mathcal{P}_0, a_{\mu v}(\vec{k})] = k_0 a_{\mu v}(\vec{k}),$$
$$[\mathcal{P}_0, a^{+\mu v}(\vec{k})] = -k_0 a^{+\mu v}(\vec{k}). \tag{55}$$

Replacing (54) in (55) we obtain at the integral equation:

$$|\vec{k}|a^{+\rho\lambda}(\vec{k'}) = \frac{1}{2}\int |\vec{k}|[a_{\mu v}(\vec{k}), a^{+\rho\lambda}(\vec{k'})]a^{+\mu v}(\vec{k})\ d^3k. \tag{56}$$

The solution of this equation is

$$\left[a_{\mu v}(\vec{k}), a^{+\rho\lambda}(\vec{k'})\right] = \left[\delta_\mu^\rho\delta_v^\lambda + \delta_v^\rho\delta_\mu^\lambda\right]\delta(\vec{k} - \vec{k'}). \tag{57}$$

As customary, in the Gupta quantization for the graviton, the physical state $|\psi >$ of the theory is defined via the equation

$$\phi_\mu^\mu|\psi >= 0. \tag{58}$$

We use now the the usual definition for the graviton's propagator

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) =< 0|T[\phi_{\mu\nu}(x)\phi^{\rho\lambda}(y)]|0 > . \tag{59}$$

Thus the propagator then turns out to be

$$\Delta_{\mu\nu}^{\rho\lambda}(x-y) = -\frac{i}{(2\pi)^4}(\delta_\mu^\rho\delta_v^\lambda + \delta_v^\rho\delta_\mu^\lambda)\int \frac{e^{ik_\mu(x^\mu - y^\mu)}}{k^2 - i0}d^4k. \tag{60}$$

The tensor field of the graviton is defined as:

$$\Phi(x) = \phi^{\rho\lambda}(x)dx_\rho \otimes dx_\lambda. \tag{61}$$

The corresponding propagator results:

$$\Delta(x-y) =< 0|T[\Phi(x)\otimes\Phi(y)]|0 > . \tag{62}$$

This is:

$$\Delta(x-y) = \Delta_{\mu\nu}^{\rho\lambda}(x-y)dx_\rho \otimes dx_\lambda \otimes dy^\mu \otimes dy^\nu. \tag{63}$$

Using (54) we can write:

$$\mathcal{P}_0 = -\frac{1}{4}\int |\vec{k}|\left[a_{\mu v}(\vec{k})a^{+\mu v}(\vec{k'}) + a^{+\mu v}(\vec{k'})a_{\mu v}(\vec{k})\right]\delta(\vec{k} - \vec{k'})d^3kd^3k'. \tag{64}$$

According (57) we get:

$$\mathcal{P}_0 = -\frac{1}{4}\int |\vec{k}| \left[2a^{+\mu v}(\vec{k}')a_{\mu v}(\vec{k}) + \delta(\vec{k}-\vec{k}')\right]\delta(\vec{k}-\vec{k}')d^3kd^3k'. \quad (65)$$

We then obtain:

$$\mathcal{P}_0 = -\frac{1}{2}\int |\vec{k}|a^{+\mu v}(\vec{k})a_{\mu v}(\vec{k})d^3k. \quad (66)$$

Here where we have used the fact that the product of two deltas with the same argument vanishes [3], i.e., $\delta(\vec{k}-\vec{k}')\delta(\vec{k}-\vec{k}') = 0$. This proves that using Ultrahyperfunctions is here equivalent to adopting the normal order in the definition of the time-component of the four-momentum

$$\mathcal{P}_0 = -\frac{1}{4}\int |\vec{k}| : \left[a_{\mu v}(\vec{k})a^{+\mu v}(\vec{k}) + a^{+\mu v}(\vec{k})a_{\mu v}(\vec{k})\right] : d^3k. \quad (67)$$

Now, we must insist on the fact that the physical state should satisfy not only Eq. (58) but also the relation (see [8–10])

$$\partial_\mu \phi^{\mu v}|\psi>= 0. \quad (68)$$

The resulting theory is similar to that obtained for QED, using the Guppta-Bleuler quantization method. This show that the obtained theory is unitary for any finite perturbative order. If we take into account the degrees of freedom of the theory, we conclude that we have only one type of free graviton ϕ^{12}.Thus we have only one type of graviton with two possible transverse polarizations. Obviously, this happens for a non-interacting theory, as remarked by Gupta.

4.1 Loss of unitarity if our constraint is not used

If we do NOT use the new constraint (58), we have

$$\mathcal{P}_0 = -\frac{1}{2}\int |\vec{k}| \left[a^{+\mu v}(\vec{k})a_{\mu v}(\vec{k}) - \frac{1}{2}a^{+\mu}_{\mu}(\vec{k})a^{v}_{v}(\vec{k})\right] d^3k. \quad (69)$$

The Feynman-Schwinger variational principle [7] now leads us to:

$$|\vec{k}|a^{+}_{\rho\lambda}(\vec{k}') = \frac{1}{2}\int |\vec{k}| \left\{a^{+\mu v}(\vec{k})[a_{\mu v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')] - \frac{1}{2}a^{+\mu}_{\mu}(\vec{k})[a^{v}_{v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')]\right\} d^3k. \quad (70)$$

The solution of this integral equation is now given by:

$$\left[a_{\mu v}(\vec{k}), a^{+}_{\rho\lambda}(\vec{k}')\right] = [\eta_{\mu\rho}\eta_{v\lambda} + \eta_{v\rho}\eta_{\mu\lambda} - \eta_{\mu v}\eta_{\rho\lambda}]\,\delta(\vec{k}-\vec{k}'). \quad (71)$$

The above is the usual graviton's quantification. The resulting theory leads to a S matrix that is not unitary [8–10,30].

5 Appendix E: The Convolution of two Lorentz Invariant Ultrahyperfunctions

We clarify that the content of this appendix has been taken from the references [5,6] in order to simplify the reading of the paper.

In [5] formula (7.34) we have obtained a conceptually simple but rather lengthy expression for the convolution of two Lorentz invariant tempered ultradistributions:

$$H_\lambda(\rho,\Lambda) = \frac{1}{8\pi^2\rho}\int_{\Gamma_1}\int_{\Gamma_2} F(\rho_1)G(\rho_2)\rho_1^\lambda\rho_2^\lambda \left\{\Theta[\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times\right.\right.$$

$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\left\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\Bigg\}+$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\Bigg\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+$$
$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\Bigg\}\Bigg\}-\frac{i}{2}\times$$
$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\Bigg\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+$$
$$\frac{(\rho_1-\rho_2)}{2}[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-$$
$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2[\ln(-\rho_1-\rho_2+\Lambda)-$$
$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\Bigg\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+$$
$$\frac{(\rho_1-\rho_2)}{2}[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-$$
$$\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)]+\rho_1[\ln(-\rho_1-\rho_2+\Lambda)-$$
$$\ln(-\rho_1-\rho_2-\Lambda)]+\rho_2[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)]\}\}\}\ d\rho_1\,d\rho_2. \tag{72}$$

This defines an ultradistribution in the variables ρ and Λ for

$$|\Im(\rho)|>\Im(\Lambda)>|\Im(\rho_1)|+|\Im(\rho_2)|.$$

Let $\mathfrak{B}$ be a vertical band contained in the complex λ-plane $\mathfrak{P}$. Integral (72) is an analytic function of λ defined in the domain $\mathfrak{B}$. Moreover, it is bounded by a power of $|\rho\Lambda|$. Then, $H_\lambda(\rho,\Lambda)$ can be analytically continued to other parts of $\mathfrak{P}$. Thus, we define

$$H(\rho)=H^{(0)}(\rho,i0^+), \tag{73}$$

$$H_\lambda(\rho, i0^+) = \sum_{-m}^{\infty} H^{(n)}(\rho, i0^+)\lambda^n. \tag{74}$$

As in the other cases, we define now

$$\{F * G\}(\rho) = H(\rho), \tag{75}$$

as the convolution of two Lorentz invariant tempered ultradistributions.
Alternatively we can use the formula obtained in [6], formula (10.1) for Ultrahyperfunctions of exponential type:

$$H_{\gamma\lambda}(\rho,\Lambda) = \frac{1}{8\pi^2\rho}\int_{\Gamma_1}\int_{\Gamma_2}[2\cosh(\gamma\rho_1)]^{-\lambda}F(\rho_1)[2\cosh(\gamma\rho_2)]^{-\lambda}G(\rho_2)$$
$$\{\Theta[\Im(\rho)]\,\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times$$
$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$
$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$
$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\left.\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$
$$\Theta[-\Im(\rho)]\,\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$
$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$
$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}+$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$
$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$
$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}\right\}-\frac{i}{2}\times$$
$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$
$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$
$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$
$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-\right.$$
$$\left.\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)\right]+\rho_2\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$
$$\left.\left.\ln(-\rho_1-\rho_2-\Lambda)\right]+\rho_1\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)\right]\right\}$$
$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$
$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right]+\right.$$
$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)-\right.$$
$$\left.\ln(-\rho_1-\rho_2+\Lambda)+\ln(-\rho_1-\rho_2-\Lambda)\right]+\rho_1\left[\ln(-\rho_1-\rho_2+\Lambda)-\right.$$
$$\left.\left.\left.\ln(-\rho_1-\rho_2-\Lambda)\right]+\rho_2\left[\ln(\rho_1+\rho_2+\Lambda)-\ln(\rho_1+\rho_2-\Lambda)\right]\right\}\right\}\right\}\ d\rho_1\ d\rho_2 \qquad (76)$$

$$|\Im(\rho)| > \Im(\Lambda) > |\Im(\rho_1)| + |\Im(\rho_2)| \; ; \; \gamma < \min\left(\frac{\pi}{2 \mid \Im(\rho_1) \mid}; \frac{\pi}{2 \mid \Im(\rho_2) \mid}\right)$$

We define

$$H(\rho) = H^{(0)}(\rho, i0^+) = H^{(0)}_\gamma(\rho, i0^+), \tag{77}$$

$$H_{\gamma\lambda}(\rho, i0^+) = \sum_{-m}^{\infty} H^{(n)}_\gamma(\rho, i0^+)\lambda^n. \tag{78}$$

If we take into account that singularities (in the variable Λ) are contained in a horizontal band of width $|\sigma_0|$ we have:

$$H_{\gamma\lambda}(\rho, i0^+) = \sum_{-m}^{\infty} H^{(n)}_{\gamma\lambda}(\rho, i\sigma)\frac{(-i\sigma)^n}{n!}, \quad \sigma > |\sigma_0|. \tag{79}$$

As in the other cases we define now

$$\{F * G\}(\rho) = H(\rho), \tag{80}$$

as the convolution of two Lorentz invariant ultradistributions of exponential type.
Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier antitransform of $H_{\gamma\lambda}(\rho, i0^+)$:

$$\hat{H}_{\gamma\lambda}(x) = \sum_{n=-m}^{\infty} \hat{H}^{(n)}_\gamma(x)\lambda^n. \tag{81}$$

If we define:

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{F_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}F(\rho)\},$$
$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{G_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}G(\rho)\}, \tag{82}$$

then

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4\hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x), \tag{83}$$

and taking into account the Laurent's developments of $\hat{f}$ and $\hat{g}$:

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}^{(n)}_\gamma(x)\lambda^n,$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}^{(n)}_\gamma(x)\lambda^n, \tag{84}$$

we can write:

$$\sum_{n=-m}^{\infty} \hat{H}^{(n)}_\gamma(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty}\left(\sum_{k=-m}^{n} \hat{f}^{(k)}_\gamma(x)\hat{g}^{(n-k)}_\gamma(x)\right)\lambda^n, \qquad (m = m_f + m_g), \tag{85}$$

and as a consequence:

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}^{(k)}_\gamma(x)\hat{g}^{(n-k)}_\gamma(x). \tag{86}$$

The Feynman propagators corresponding to a massless particle F and a massive particle G are, respectively, the following ultrahyperfunctions:

$$F(\rho) = -\Theta[-\Im(\rho)]\rho^{-1},$$

$$G(\rho) = -\Theta[-\Im(\rho)](\rho + m^2)^{-1}, \tag{87}$$

where ρ is the complex variable, such that on the real axis one has $\rho = k_1^2 + k_2^2 + k_3^2 - k_0^2$. On the real axis, the previously defined propagators are given by:

$$\begin{aligned} f(\rho) &= F(\rho + i0) - F(\rho - i0) = (\rho - i0)^{-1}, \\ g(\rho) &= G(\rho + i0) - G(\rho - i0) = (\rho + m^2 - i0)^{-1}. \end{aligned} \tag{88}$$

These are the usual expressions for Feynman propagators.
Consider first the convolution of two massless propagators. We use (88), since here the corresponding ultrahyperfunctions do not have singularities in the complex plane. We obtain from (72) a simplified expression for the convolution:

$$h_\lambda(\rho) = \frac{\pi}{2\rho} \iint_{-\infty}^{\infty} (\rho_1 - i0)^{\lambda-1} (\rho_2 - i0)^{\lambda-1} \left[(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2\right]_+^{\frac{1}{2}} \, d\rho_1 \, d\rho_2. \tag{89}$$

This expression is nothing other than the usual convolution:

$$h_\lambda(\rho) = (\rho - i0)^{\lambda-1} * (\rho - i0)^{\lambda-1}. \tag{90}$$

6 Appendix F: A Mathematical Proof

According to the Ultrahyperfunctions theory we can write:

$$\oint_\Gamma \ln(a-z)\phi(z)dz = \int_{-\infty}^{\infty} [\ln(a-x-i0) - \ln(a-x+i0)]\phi(x)dx = -2i\pi \int_{-\infty}^{\infty} H(x-a)\phi(x)dx. \tag{91}$$

So we have the correspondence:

$$-\frac{1}{2\pi i} \ln(a-z) \longleftarrow H(x-a). \tag{92}$$

Using now the Dirac formula for Ultrahyperfunctions we obtain:

$$-\frac{1}{2\pi i} \ln(a-z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{H(x-a)}{x-z} dx = \frac{1}{2\pi i} \int_a^{\infty} \frac{1}{x-z} dx. \tag{93}$$

Thus:

$$\ln(a-z) = -\int_a^{\infty} \frac{1}{x-z} dx. \tag{94}$$

We then have for $a > 0$

$$\ln a = -\int_a^{\infty} \frac{1}{x} dx. \tag{95}$$

According to the result obtained by Guelfand in [33]

$$\int_0^{\infty} \frac{1}{x} dx = 0, \tag{96}$$

and therefore:

$$\ln a = \int_0^{a} \frac{1}{x} dx. \tag{97}$$