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Nonextensive Statistical Mechanics and Black Hole Thermodynamics: Tsallis and Kaniadakis Entropies

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Abstract. We study the impact of nonextensive entropy on the thermodynamics of various black hole configurations, utilizing both Tsallis and Kaniadakis statistical frameworks. Additionally, we explore the stability of these black holes using the framework of nonextensivity. Our analysis reveals that the nonextensive Kaniadakis entropy does not result in any stability for the black holes. In contrast, the nonextensive Tsallis entropy ensures the stability of various black hole configurations.

Keywords: Classical Statistical Mechanics; General Relativity; Nonextensive Entropies.

1 Introduction

The Hawking evaporation process causes black holes to release radiation, which is how the ideas of Bekenstein entropy [1] and Hawking temperature [2] related to the black hole horizon were developed. Within the subject of quantum field theory, the evaporation process of black holes is interesting because it seems to indicate nonunitary development, which leads to the well-known problem of the information loss dilemma [3]. Black hole thermodynamics laws [4–6] are comparable to standard thermodynamic laws, and in this setting, black holes behave like thermodynamic entities. Numerous cosmological and gravitational scenarios have made substantial use of the thermodynamics of black holes [7–12].

Entropy is a notion that measures how difficult it is for an outsider to discern a system's fundamental structure. It encapsulates the macroscopic features derived from quantum statistical mechanics that govern the behavior of quantum microstates. Bekenstein entropy in the context of black holes is exclusively defined by Hawking's area theorem and is not determined by quantum statistical mechanics [13]. Thus, to properly comprehend the formation of this entropy and, in the case of black holes, the nature of microstates, a thorough theory of quantum gravity is required. As such, we use the Bekenstein entropy definition for black holes. The Bekenstein entropy S_B and the Hawking temperature T_H of a Schwarzschild black hole with mass M are to the form of $T_H \propto M^{-1}$ and $S_B \propto A$ where the event horizon's area A of the black hole is described as $A = 4\pi r_h^2$ where $r_h = 2M$ is the event horizon or Schwarzschild radius. Notably, when the mass M gets closer to zero during the Hawking evaporation process, the Hawking temperature T_B diverges to infinity, and the Bekenstein entropy S_B approaches zero. These characteristics point to a black hole completely evaporating due to the emission of Hawking radiation.

The fundamental principle of statistical mechanics and Gibbs thermodynamics is the extensive nature of entropy. However, the rejection of this premise gives rise to nonextensive statistical mechanics, such as Tsallis nonextensive statistical mechanics [14–17]. The assumption of neglecting long-range forces between thermodynamic sub-systems is associated with the extensive nature of entropy. When the system's magnitude surpasses the range of interaction between its constituent parts, Gibbs thermodynamics neglects these long-range forces. Entropy increases with system size, and the overall entropy of a composite system is equal to the sum of the entropies of the constituent subsystems. It is important to note that Bekenstein entropy, regarded as a nonextensive quantity, scales with area when considering a black hole as a $(3 + 1)$-dimensional entity [18–22]. Furthermore, Bekenstein entropy is nonadditive due to the area scaling. As a result, the application of statistical mechanics or Gibbs thermodynamics to research the thermodynamics of black holes may not be appropriate. To understand the nonextensive and nonadditive character of Bekenstein entropy, various extensions [14,23–25] of normal Gibbs thermodynamics have been applied to black holes and cosmological horizons [26–34]. It is a well-established fact that quantum corrections affect the entropy of black holes. For example, exponential corrections are considered when counting microstates for quantum states located on the horizon. Recently, researchers have explored the modified thermodynamics of certain black holes in connection with these exponential corrections [35–37]. Non-perturbative corrections in the thermodynamics of static dirty black holes are examined in [38]. Additionally, the characteristics of black hole entropy are analyzed through the framework of general conformal field theory in [39]. Notable approaches include the black hole entropy definition by Tsallis and Cirto [18], which makes the black entropy comprehensive and consistent with the Legendre structure. Another applicable notion of entropy for black holes and cosmic horizons is Rényi entropy [40], which is nonextensive but additive (by assumption) as a measure of entanglement. Other proposed nonextensive forms of entropy include Barrow entropy [25] based on a hypothetical fractal structure of the black hole horizon, Kaniadakis entropy [23] inspired by Lorentz group transformations, and Sharma-Mittal entropy as a generalization of Rényi entropy.

The classical thermodynamics of large-scale, macroscopic objects has a strong foundation in the principles of statistical physics. In this formalism, the macroscopic characteristics of a system can be deduced directly from the microscopic description of that system.

A key assumption made in this framework is that long-range interactions between the system's components can be neglected. This leads to the establishment of well-defined local extensive parameters, such as temperature and pressure. Additionally, the entropy function of the system is obtained using the additive Boltzmann-Gibbs formula. However, for strongly gravitating systems such as black holes, the assumption of negligible long-range interactions is no longer valid. In these cases, the usual local extensive parameters used in classical thermodynamics are not suitable. This challenges the applicability of the Boltzmann-Gibbs definition of additive entropy for strongly gravitating systems.

Bekenstein entropy has been shown to be a nonextensive quantity for such systems. This has led to the development of studies aimed at understanding black hole entropy and the characteristic features of their evaporation, in the context of nonextensive thermodynamics. Rényi entropy and Tsallis entropy have been proposed as natural entropies associated with nonextensive thermodynamics, offering different generalizations compared to the Boltzmann-Gibbs entropy. The purpose of this study is to investigate how nonextensive entropies affect thermodynamics of Schwarzschild, Reissner-Nordström, and Kerr black holes of the general relativity in Tsallis and Kaniadakis nonextensive statistics. The stability of these black hole within the nonextensive statistical frameworks will be investigated. Throughout this study we assume the units $c = \hbar = G = k_B = 1$ is assumed in which c is the speed of light, $\hbar$ is the reduced Planck's constant, G is the Newtonian gravitational constant, and k_B is the Boltzmann constant. All the above information gives us the motivation to arrange our paper as follows:

The paper is organized as follows: In Section 2 we give some summary of the extensive thermodynamic quantities of the Schwarzschild, Reissner-Nordström, and Kerr black holes . In Section 3 we provide a brief description of nonextensive Tsallis and Kaniadakis entropies. Using the previously described nonextensive entropies, we examine the thermodynamics of these black holes in Section 4. In Section 5, we wrap up and go over our results.

2 Review on Black Hole Solutions and Thermodynamics

In this section, we provide a concise overview of the key metrics associated with Schwarzschild, Reissner-Nordström, and Kerr black holes. Additionally, we explore their extensive thermodynamic properties. To do this, we implement some valuable reviews and textbooks, e.g., Refs [41–46].

2.1 Schwarzschild Black Hole and the Corresponding Extensive Thermodynamics

The well-known Schwarzschild black hole metric is found as follows

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1}$$

where M is the black hole's mass. The event horizon of the Schwarzschild black hole can be determined by the condition $g_{00} = 0$, which leads to the following relation

$$r_h = 2M\,, \tag{2}$$

which is also known as Schwarzschild radius. For a Schwarzschild black hole, the first law of black hole thermodynamics can be expressed in the following form

$$dM = \frac{\kappa}{8\pi} dA \equiv T_H dS \,, \tag{3}$$

where $\kappa = (4M)^{-1}$ is the surface gravity on the event horizon of the black hole, A is the area of the event horizon, $T_H = \kappa/2\pi$ is the Hawking temperature, and $S = A/4$ is the black hole's entropy. Therefore, one can write

$$\begin{aligned} &A = 4\pi r_h^2 = 16\pi M^2, \\ &S = 4\pi M^2 \,. \end{aligned} \tag{4}$$

Using equations (3) and (4), we obtain

$$T_H = \frac{dM}{dS} = \frac{1}{8\pi M} \,, \tag{5}$$

which is compatible with $T_H = \kappa/2\pi$.
Alternatively, the heat capacity can be expressed as

$$C = \frac{dM}{dT_H} = -\frac{(S'(M))^2}{S''(M)} \,, \tag{6}$$

where a prime represents a derivative with respect to M. As a result, the heat capacity of the Schwarzschild black hole can be determined as

$$C = -\frac{1}{8\pi T^2} = -8\pi M^2 \,. \tag{7}$$

It is clear that the heat capacity of the Schwarzschild black hole is negative for all mass values, indicating that it exhibits thermodynamic instability. Furthermore, the Gibbs free energy for the Schwarzschild black hole can be expressed as

$$G = M - TS \,. \tag{8}$$

Hence, for the Schwarzschild black hole, the Gibbs free energy is given by the following equation

$$G = \frac{M}{2} \,, \tag{9}$$

which is always positive.

2.2 Reissner-Nordström Black Hole and the Corresponding Extensive Thermodynamics

The Reissner-Nordström black hole metric background is described by the following equation

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,, \tag{10}$$

where Q represents the electric charge of the source. The horizons of the Reissner-Nordström black hole, when $M > |Q|$, are the roots of $g_{00} = 0$, which are given by

$$r_\pm = M \pm \sqrt{M^2 - Q^2} \,, \tag{11}$$

where r_+ is the event horizon and r_- is the Cauchy horizon of the Reissner-Nordström black hole, assuming the condition $M^2 > Q^2$ is satisfied. It is also noted that the product of these two horizons is equal to Q^2.

For the Reissner-Nordström black hole, the first law of black hole thermodynamics is described by the following equation

$$dM = \frac{\kappa}{8\pi} dA + \Phi dQ \equiv T_H dS + \Phi dQ \,, \tag{12}$$

where

$$\kappa = \frac{2\left(M\sqrt{M^2 - Q^2} + M^2 - Q^2\right)}{\left(\sqrt{M^2 - Q^2} + M\right)^3},$$

represents the surface gravity of the event horizon of the Reissner-Nordström black hole whereas $\Phi = Q/r_+$ denotes the electrostatic potential of a test charge crossing the horizon. The entropy of the Reissner-Nordström black hole is given by

$$S = \pi\left(M + \sqrt{M^2 - Q^2}\right)^2 . \tag{13}$$

Using equations (12) and (13), we find

$$T_H = \frac{\partial M}{\partial S} = \frac{\left(M^2 - Q^2 + M\sqrt{M^2 - Q^2}\right)}{\pi\left(M + \sqrt{M^2 - Q^2}\right)^3} \,, \tag{14}$$

where is the same as $T_H = \kappa/2\pi$. Using (6), the heat capacity of the Reissner-Nordström black hole is obtained as follows

$$C = \frac{2\pi\sqrt{M^2 - Q^2}\left(M + \sqrt{M^2 - Q^2}\right)^2}{M - 2\sqrt{M^2 - Q^2}} \,. \tag{15}$$

From the above equation, one can conclude that the Reissner-Nordström black hole is thermodynamically stable if the conditions $M^2 > Q^2$ and

$$Q^2 < \left(M + \sqrt{M^2 - Q^2}\right)^2 < 3Q^2,$$

is fulfilled. Based on equation (8), we find the Gibbs free energy of the Reissner-Nordström black hole one in the following form

$$G = M - \frac{1}{2}\sqrt{M^2 - Q^2} \,. \tag{16}$$

2.3 Rotating Kerr Black Hole and the Corresponding Extensive Thermodynamics

The metric of Kerr black hole is described by the following equation

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} dt d\phi \\ &+ \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2a^2 Mr\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \,, \end{aligned} \tag{17}$$

where $J = aM$ represents the angular momentum of the source and we have defined

$$\begin{aligned} \rho^2 &\equiv r^2 + a^2 \cos^2\theta\,, \\ \Delta &\equiv r^2 + a^2 - 2Mr\,. \end{aligned} \tag{18}$$

By solving the equation $\Delta = 0$, one can find the horizons of the Kerr black hole when $M > a$ as

$$\begin{aligned} r_{out} &= M + \sqrt{M^2 - a^2}\,, \\ r_{in} &= M - \sqrt{M^2 - a^2}\,, \end{aligned} \tag{19}$$

where r_{out} is the event horizon and r_{in} is the inner horizon of the Kerr black hole. It should be noted that $r_{out} r_{in} = a^2$.

The first law of black hole thermodynamics corresponding to the Kerr black hole is given by

$$dM = \frac{\kappa}{4\pi} dA + \Omega dJ \equiv T_H dS + \Omega dJ\,, \tag{20}$$

where we have

$$\begin{aligned} \kappa &= \frac{\sqrt{M^2 - a^2}}{a^2 + \left(\sqrt{M^2 - a^2} + M\right)^2}\,, \\ \Omega &= \frac{a}{\left(\sqrt{M^2 - a^2} + M\right)^2 + a^2}\,, \end{aligned} \tag{21}$$

in which Ω is the angular velocity of the Kerr black hole. The area A of the event horizon Kerr black hole can be calculated as follows

$$A = \int_0^{2\pi}\int_0^{\pi} (r_{out}^2 + a^2)\sin\phi\, d\phi = 4\pi(r_{out}^2 + a^2) = 4\pi\left(\left(M + \sqrt{M^2 - a^2}\right)^2 + a^2\right). \tag{22}$$

From the above equation,the entropy is obtained as

$$S = \pi\left(\left(M + \sqrt{M^2 - a^2}\right)^2 + a^2\right). \tag{23}$$

Using equations (20) and (23), we obtain

$$T_H = \frac{\partial M}{\partial S} = \frac{\sqrt{M^2 - a^2}}{2\pi\left(a^2 + \left(M + \sqrt{M^2 - a^2}\right)^2\right)}\,, \tag{24}$$

as the temperature of the Kerr black hole, which has the same result as $T_H = \kappa/2\pi$.

From (6), the heat capacity of the Kerr black hole can be found as follows

$$C = \frac{2\pi\sqrt{M^2 - a^2}\left(M + \sqrt{M^2 - a^2}\right)^2}{M - 2\sqrt{M^2 - a^2}}\,. \tag{25}$$

From equation (25), one can verify that the Kerr black hole is thermodynamically stable if the condition $M > a$ and

$$2aM < \left(\sqrt{M^2 - a^2} + M\right)^2 + a^2 < 2aM\sqrt{2\sqrt{3} + 3},$$

is satisfied. Based on equation (8), we find

$$G = M - \frac{1}{2}\sqrt{M^2 - a^2}\,, \tag{26}$$

which is the Gibbs free energy of the Kerr black hole.

3 Introduction to Nonextensive Entropies

In this section, we provide a concise overview of the nonextensive Tsallis and Kaniadakis entropies, along with their corresponding mathematical formulations. Here, we generally follow Refs. [47–51]

3.1 Nonextensive Tsallis Entropy

Entropy is a fundamental concept within the framework of Gibbs thermodynamics, also known as statistical mechanics. In this formalism, entropy is a large quantity and follows the additive composition rule. A key assumption made in Boltzmann-Gibbs statistical mechanics is the disregard of long-range forces. This means that the interactions between the various components of the system are considered to be predominantly local, without significant long-range effects. On the other hand, certain physical systems might not be well-suited to Gibbs thermodynamics [52], as they are subject to long-range forces. Interestingly, long-range forces have a substantial influence on some self-gravitating phenomena, such as black holes, therefore Gibbs thermodynamics is not suitable for them. To solve this problem, Constantino Tsallis expanded the traditional Gibbs entropy for nonextensive systems [14,52]. One of the first ideas to expand on Boltzmann-Gibbs entropy was Tsallis entropy, or ST. It is defined by the following equation

$$S_T = -\sum_i [p(i)]^q \ln_q p(i) \,, \tag{27}$$

where the probability distribution $p(i)$ is specified on a collection of microstates W, and we take it to be positive. The degree of nonextensivity is determined by the parameter q. The q-logarithmic function is defined as

$$\ln_q p = \frac{p^{1-q} - 1}{1 - q} \,. \tag{28}$$

Furthermore, in the limit $q \to 1$, the Tsallis entropy expressed in equation (28) converges to the Boltzmann-Gibbs entropy S_G.

Through a recent investigation into the relationship between Tsallis and Boltzmann-Gibbs statistics [49], a unique entropy for black holes has been established as follows

$$S_T = \frac{1}{1 - q} \left(\exp[S(1 - q)] - 1\right) \,. \tag{29}$$

where S represents the Bekenstein entropy, and q is a free, unknown parameter as given in equation (27). To be precise, equation (29) represents a form of Tsallis entropy for black holes. For $q \to 1$, the Bekenstein entropy S will recovered from equation (29).

3.2 Nonextensive Kaniadakis Entropy

The Kaniadakis entropy is a form of nonextensive entropy that arises from the Lorentz transformation in special relativity. This type of nonextensive entropy has only a single parameter [23,34]. By introducing a new parameter K into the conventional Boltzmann-Gibbs entropy and connecting it to the relativistic domain, the dimensionless rest energy of the component elements in a multi-body relativistic system can be represented. This modification results in a deformation or alteration of the Boltzmann-Gibbs entropy. The Kaniadakis entropy is defined by the following relation

$$S_K = \log_K \Omega \,, \tag{30}$$

where

$$\log_K \Omega = \frac{\Omega^K - \Omega^{-K}}{2K} \,. \tag{31}$$

Note that the Boltzmann-Gibbs entropy is recovered in the limit of $K \to 0$. Equation (30) can be rewritten in the following form for black holes

$$S_K = \frac{1}{K} \sinh[KS] \,, \tag{32}$$

where S represents the Bekenstein entropy. For $K \to 0$, equation (32) can be recovered to the Bekenstein entropy.

4 Application of Nonextensive Entropies to Black Hole Thermodynamics

In this section, we will apply the nonextensive Tsallis and Kaniadakis entropies to the thermodynamic analysis of three types of black holes: Schwarzschild, Reissner-Nordström, and Kerr black holes.

4.1 Nonextensive Tsallis Entropy

4.1.1 Schwarzschild Black Hole

To apply the nonextensive Tsallis entropy to the Schwarzschild black hole, we insert equation (4) into equation (29) to obtain the corresponding nonextensive Tsallis entropy for the event horizon of the black hole as

$$S_T = \frac{1}{1-q} \left(\exp[4\pi M^2(1-q)] - 1 \right) \,. \tag{33}$$

Figure 1 shows the nonextensive Tsallis entropy to the Schwarzschild black hole for different values of q versus M. The values of q are based on the stability of the black hole as seen in the following equation (35). From Figure 1, we see that decreasing the value q leads to increase the nonextensive Tsallis entropy. Also, the ordinary Bekenstein entropy of Schwarzschild black hole associated with $q = 1$ diverges by increasing M.

Form equation (33) together with equation (5), we can find the Hawking temperature of the Schwarzschild black hole influenced by nonextensive Tsallis entropy which is given by

$$T_H = \frac{\exp[4\pi M^2(q-1)]}{8\pi M} \,. \tag{34}$$

For $q \to 1$, the ordinary Hawking temperature for the black hole will recover from equation (34). Comparing equation (34) with $T_H = \kappa/2\pi$, we see that nonextensive Tsallis statistics even affect on the gravitational theory outcomes such that surface gravity becomes

$$\kappa = \frac{\exp[4\pi M^2(q-1)]}{4M} \,,$$

which depends on q parameter. Figure 2 demonstrates the Hawking temperature inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q. The values of q are based on the stability of the black hole as seen in the following equation (35). Figure 2 shows us that for $M \to 0$, the Hawking temperature inspired by nonextensive

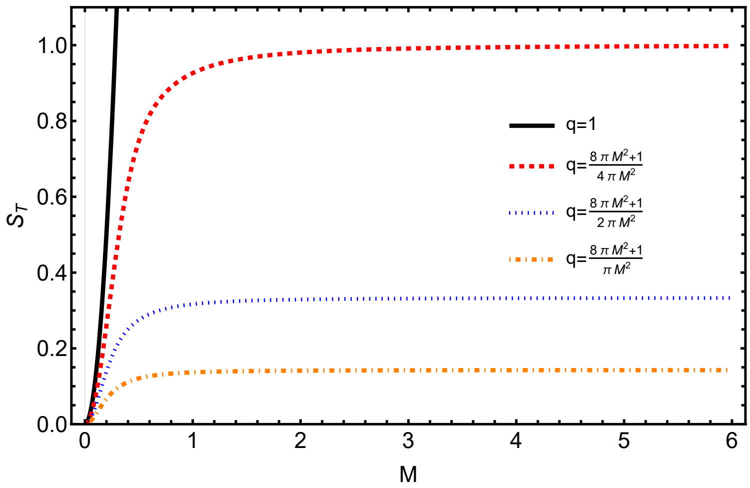

Figure 1: The illustration of the nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q versus M. The $q = 1$ case is for the ordinary Bekenstein entropy of Schwarzschild black hole.

Tsallis entropy like ordinary Hawking temperature diverges. Also, increasing q results in increasing the Hawking temperature inspired by the nonextensive Tsallis entropy. Despite the Schwarzschild black hole having the ordinary Hawking temperature, which vanishes as the black hole mass $M \to \infty$, the Hawking temperature inspired by the nonextensive Tsallis entropy of the black hole actually diverges as $M \to \infty$.

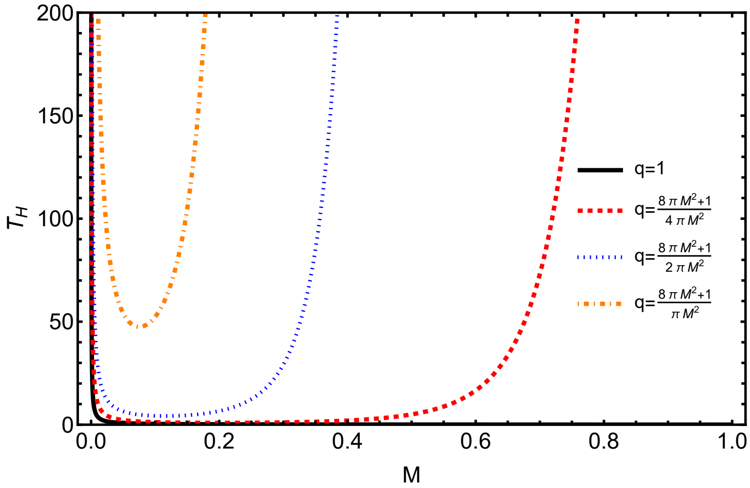

Figure 2: The illustration of the Hawking temperature inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q versus M. The $q = 1$ case is for the ordinary Bekenstein entropy of Schwarzschild black hole.

From equations (6) and (33), we can obtain the relevant expression for the heat capacity inspired by nonextensive Tsallis entropy as follows

$$C = \frac{8\pi M^2 \exp[4\pi M^2(1-q)]}{8\pi M^2(q-1)-1}\,. \tag{35}$$

When $q \to 1$, the ordinary heat capacity for the black hole will recover from equation (35). One can check that if the condition

$$q > \frac{8\pi M^2 + 1}{8\pi M^2},$$

is satisfied, the heat capacity inspired by nonextensive Tsallis entropy (35) is positive and therefore, the black hole is thermodynamically stable. Figure 3 shows the heat capacity inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for different values of q versus M. The values of q are based on the stability of the black hole as seen in equation (35). The stability of the Schwarzschild black hole with nonextensive Tsallis entropy by choosing the specific values of q is clear from Figure 3 while the Schwarzschild black hole with ordinary statistics is unstable as seen in this figure. From Figure 3 we see that decreasing q leads to increase the heat capacity inspired by nonextensive Tsallis entropy of the Schwarzschild black hole.

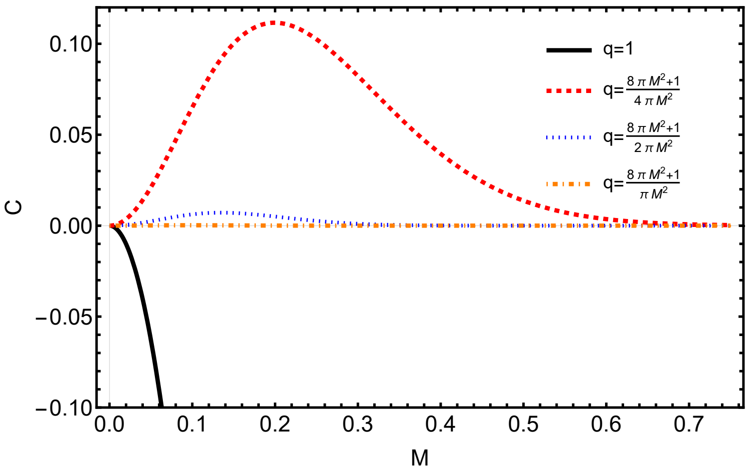

Figure 3: The illustration of the heat capacity inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q versus M. The $q = 1$ case is for the ordinary heat capacity of Schwarzschild black hole.

From equation (8), the Gibbs free energy is affected by nonextensive Tsallis entropy which is obtained by

$$G = \frac{1 + 8\pi M^2(q-1) - \exp[4\pi M^2(q-1)]}{8\pi M(q-1)}\,, \tag{36}$$

which again in the limit of $q \to 1$, reduces to the ordinary Gibbs free energy for the black hole. Figure 4 shows the Gibbs free energy inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q versus M. From this figure, we see that the Gibbs free energy inspired by nonextensive Tsallis entropy of the Schwarzschild black hole is negative while the ordinary the Gibbs free energy of the Schwarzschild black hole is positive. Figure 4 shows us that decreasing q results in increasing Gibbs free energy of the Schwarzschild black hole.

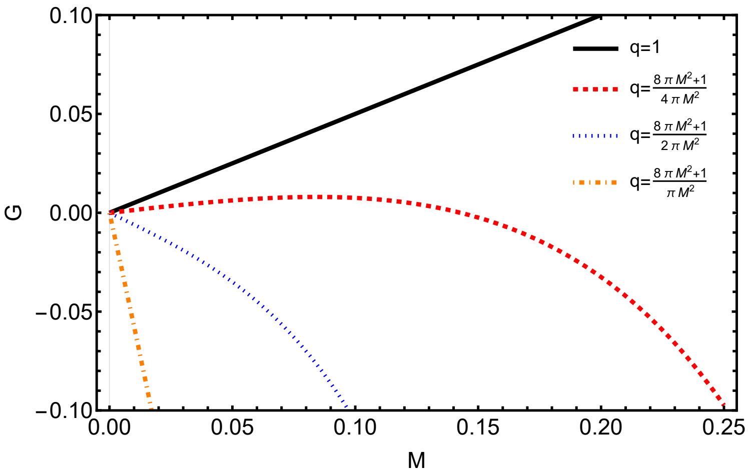

Figure 4: The illustration of the Gibbs free energy inspired by nonextensive Tsallis entropy of the Schwarzschild black hole for various values of q versus M. The $q = 1$ case is for the ordinary Gibbs free energy of Schwarzschild black hole.

4.1.2 Reissner-Nordström Black Hole

Here we want to apply the nonextensive Tsallis entropy to the Reissner-Nordström black hole. To do this, we apply equation (13) to equation (29) to gain

$$S_T = \frac{\exp[\pi(1-q)\left(M+\sqrt{M^2-Q^2}\right)^2]-1}{1-q}\,, \tag{37}$$

which is the nonextensive Tsallis entropy of the Reissner-Nordström black hole. Figure 5 illustrates the nonextensive Tsallis entropy of the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The values of q are determined based on the stability of the black hole as indicated in equation (39). From Figure 5, it is evident that increasing q results in a decrease in the nonextensive Tsallis entropy of the black hole. Furthermore, the case of $q = 1$, which is associated with the ordinary Bekenstein entropy of the Reissner-Nordström black hole, exhibits smaller values than the nonextensive Tsallis entropy of the black hole.

Now, equations (5) and (37) give us the Hawking temperature of the Reissner-Nordström black hole within the nonextensive Tsallis entropy, so we have

$$T_H = \frac{\sqrt{M^2-Q^2}\exp[\pi(q-1)\left(M+\sqrt{M^2-Q^2}\right)^2]}{2\pi\left(M+\sqrt{M^2-Q^2}\right)^2}\,. \tag{38}$$

In the limit of $q \to 1$, equation (34) reduces to the ordinary Hawking temperature for the black hole. Figure 6 depicts the Hawking temperature influenced by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The values of q are determined based on the stability of the black hole as indicated in equation (39). From Figure 6, it is evident that increasing q results in an increase in the value of the Hawking temperature influenced by the nonextensive Tsallis entropy for the Reissner-Nordström black hole. Additionally, we observe that the case of $q = 1$, which is

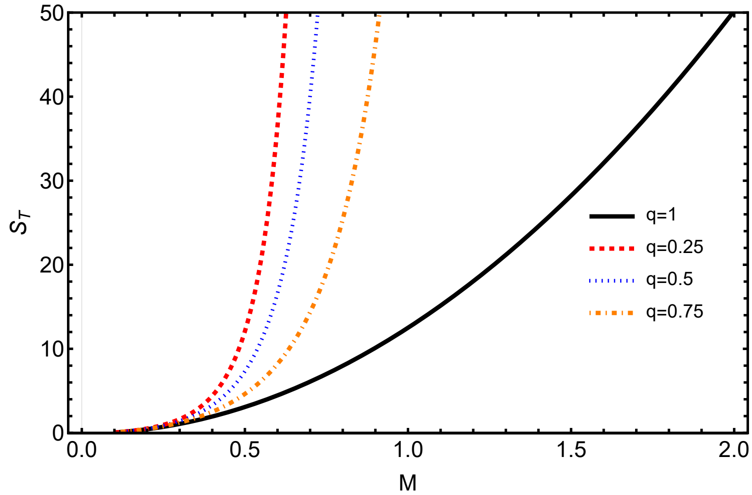

Figure 5: The illustration of the nonextensive Tsallis entropy of the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The $q = 1$ case is for the ordinary Bekenstein entropy of the Reissner-Nordström black hole.

associated with the ordinary Hawking temperature for the Reissner-Nordström black hole, has much larger values than the cases related to the Hawking temperature influenced by the nonextensive Tsallis entropy for the black hole.

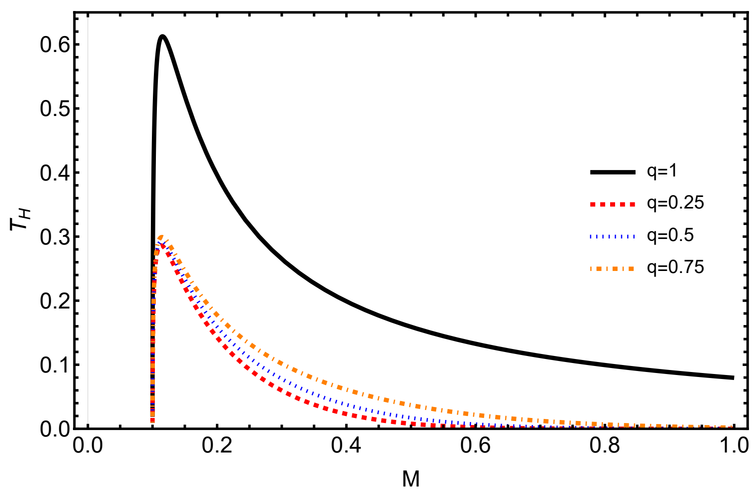

Figure 6: The illustration of the Hawking temperature inspired by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The $q = 1$ case is for the ordinary Hawking temperature of the Reissner-Nordström black hole.

Equations (6) and (37) together yield the heat capacity inspired by nonextensive Tsallis entropy for the Reissner-Nordström black hole as follows

$$C = \frac{2\pi\sqrt{M^2-Q^2}\left(M+\sqrt{M^2-Q^2}\right)^2 \exp[-\pi(q-1)\left(M+\sqrt{M^2-Q^2}\right)^2]}{4\pi M^3(q-1)+4\pi M^2(q-1)\sqrt{M^2-Q^2}-2\sqrt{M^2-Q^2}\left(\pi(q-1)Q^2+1\right)-4\pi M(q-1)Q^2+M}. \tag{39}$$

For $q \to 1$, the ordinary heat capacity for the black hole will gain from equation (39). Satisfying the conditions $q < 1$, $M^2 > Q^2$ and

$$Q^2 < \left(M+\sqrt{M^2-Q^2}\right)^2 < 3Q^2,$$

leads to positivity of the heat capacity inspired by nonextensive Tsallis entropy (39), which means that the black hole can be thermodynamically stable. Figure 7 illustrates the heat capacity influenced by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The values of q are determined based on the stability of the black hole as indicated in equation (39). From Figure 7, it is evident that the heat capacity influenced by the nonextensive Tsallis entropy and the ordinary heat capacity of the Reissner-Nordström black hole are positive under the above conditions. This indicates that both the ordinary Reissner-Nordström black hole and the one affected by the nonextensive Tsallis entropy are thermodynamically stable under the aforementioned conditions. Additionally, we observe that increasing the values of q leads to a reduction in the values of the heat capacity.

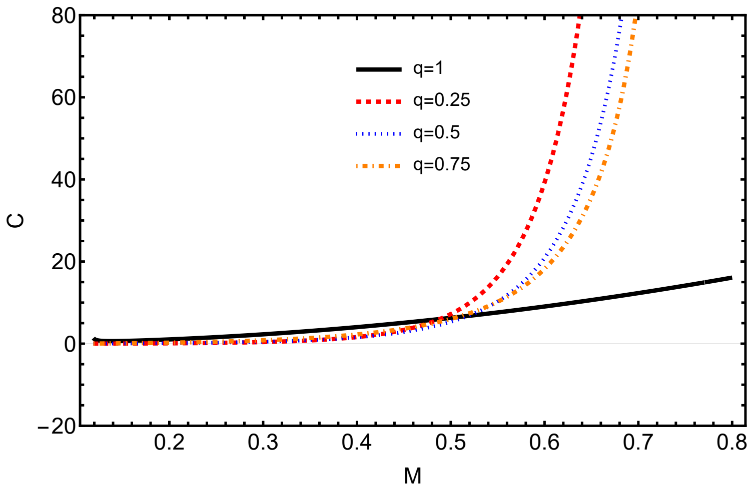

Figure 7: The illustration of the heat capacity inspired by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The $q = 1$ case is for the ordinary heat capacity of the Reissner-Nordström black hole.

Equations (8), (37), and (38) result in the Gibbs free energy affected by nonextensive Tsallis entropy for the Reissner-Nordström black hole as follows

$$G = M - \frac{\sqrt{M^2-Q^2}\left(\exp[\pi(q-1)\left(M+\sqrt{M^2-Q^2}\right)^2]-1\right)}{2\pi(q-1)\left(M+\sqrt{M^2-Q^2}\right)^2}, \tag{40}$$

Again in the limit $q \to 1$, equation (40) reduces to the ordinary Gibbs free energy for the Reissner-Nordström black hole. Figure 8 illustrates the Gibbs free energy influenced by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The values of q are determined based on the stability of the black hole as indicated in equation (39). It is evident from Figure 8 that increasing the value of q leads to a decrease in the Gibbs free energy influenced by the nonextensive Tsallis entropy for the Reissner-Nordström black hole.

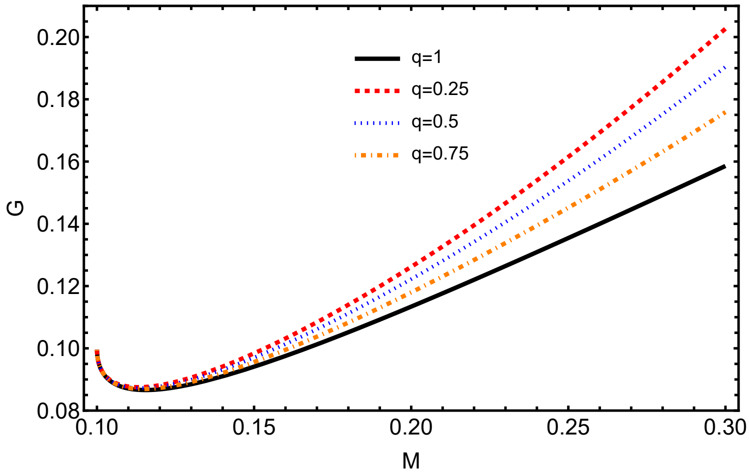

Figure 8: The illustration of the Gibbs free energy inspired by the nonextensive Tsallis entropy for the Reissner-Nordström black hole for various values of q versus M with $Q = 0.1$. The $q = 1$ case is for the ordinary Gibbs free energy of the Reissner-Nordström black hole.

4.1.3 Kerr Black Hole

Now we apply the nonextensive Tsallis entropy to the Kerr black hole by inserting equation (22) to equation (29) so we find

$$S_T = \frac{1 - \exp[-2\pi M(q-1)\left(M + \sqrt{M^2 - a^2}\right)]}{q-1} \,. \tag{41}$$

This is the nonextensive Tsallis entropy of the Kerr black hole. Figure 9 illustrates the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The values of q are determined based on the stability of the black hole as indicated in equation (43). From Figure 9, it is evident that increasing the value of q leads to a decrease in the nonextensive Tsallis entropy. Additionally, the ordinary Bekenstein entropy of the Kerr black hole associated with $q = 1$ is much smaller than the nonextensive Tsallis entropy of the Kerr black hole.

From equations (5) and (41) one achieve to the Hawking temperature of the Kerr black hole in the nonextensive Tsallis entropy framework as

$$T_H = \frac{\sqrt{M^2 - a^2}\, \exp[2\pi M(q-1)\left(M + \sqrt{M^2 - a^2}\right)]}{2\pi \left(M + \sqrt{M^2 - a^2}\right)^2} \,. \tag{42}$$

Considering $q \to 1$, equation (42) reduces to the ordinary Hawking temperature for the Kerr black hole. Figure 10 depicts the Hawking temperature influenced by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The

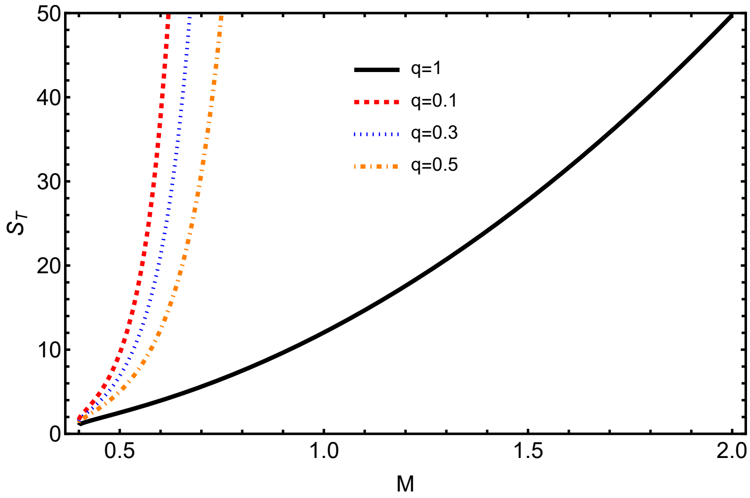

Figure 9: The illustration of the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The $q = 1$ case is for the ordinary Bekenstein entropy of the Kerr black hole.

values of q are determined based on the stability of the black hole as indicated in equation (43). Increasing the value of q leads to an increase in the Hawking temperature influenced by the nonextensive Tsallis entropy for the Kerr black hole, as observed in Figure 10, while the ordinary Hawking temperature for the Kerr black hole with $q = 1$ is much larger than the corresponding temperature affected by nonextensive Tsallis entropy.

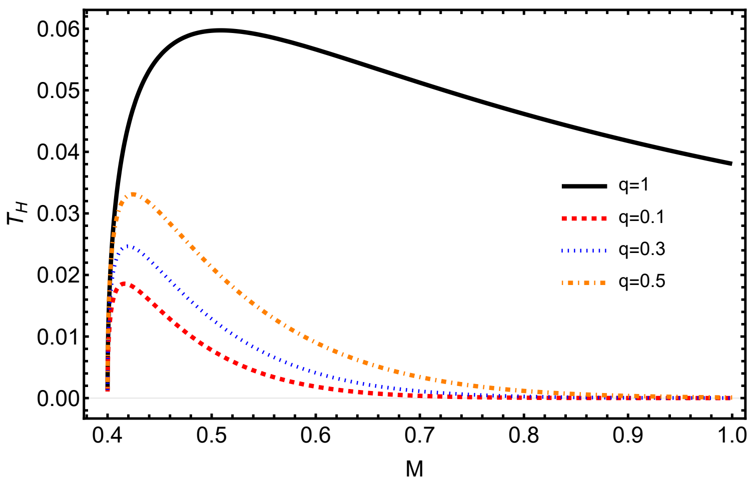

Figure 10: The illustration of the Hawking temperature inspired by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The $q = 1$ case is for the ordinary Hawking temperature of the Kerr black hole.

Employing equations (6) and (41) will result in the heat capacity inspired by nonextensive

Tsallis entropy for the Kerr black hole as

$$C = \frac{2\pi\sqrt{M^2-a^2}\left(M+\sqrt{M^2-a^2}\right)^2 \exp[-2\pi M(q-1)\left(M+\sqrt{M^2-a^2}\right)]}{4\pi M^2(q-1)\sqrt{M^2-a^2} - 2\sqrt{M^2-a^2}\left(\pi a^2(q-1)+1\right) - 4\pi a^2 M(q-1) + 4\pi M^3(q-1) + M}. \quad (43)$$

By the condition $q \to 1$, the ordinary heat capacity for the Kerr black hole will be achieved from equation (43). Based on $q < 1$, $M > a$ and

$$2aM < \left(\sqrt{M^2-a^2}+M\right)^2 + a^2 < 2aM\sqrt{2\sqrt{3}+3},$$

one can see positivity of the heat capacity inspired by nonextensive Tsallis entropy (43). This shows us that the black hole can be thermodynamically stable. Figure 11 illustrates the heat capacity influenced by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The values of q are determined based on the stability of the black hole as indicated in equation (43). Figure 11 shows that in the selected region respecting the aforementioned conditions, the heat capacity influenced by the nonextensive Tsallis entropy and the ordinary heat capacity of the Kerr black hole is positive, which results in the stability of the black hole in this parameter space. Increasing the value of q leads to a reduction in the heat capacity influenced by the nonextensive Tsallis entropy for the Kerr black hole, as observed in Figure 11.

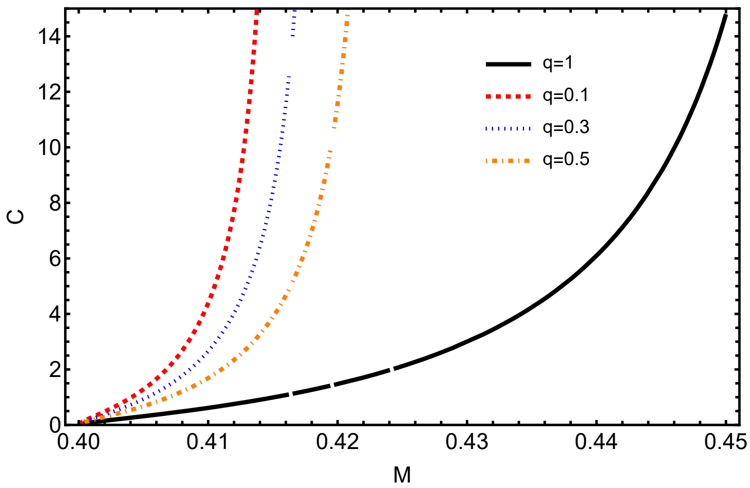

Figure 11: The illustration of the heat capacity inspired by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The $q = 1$ case is for the ordinary heat capacity of the Kerr black hole.

From equations (8), (41), and (42) we can find the Gibbs free energy affected by nonextensive Tsallis entropy for the Kerr black hole as follows

$$G = M - \frac{\sqrt{M^2-a^2}\left(\exp[2\pi M(q-1)\left(M+\sqrt{M^2-a^2}\right)]-1\right)}{2\pi(q-1)\left(M+\sqrt{M^2-a^2}\right)^2}, \quad (44)$$

Again with $q \to 1$, equation (44) reduces to the ordinary Gibbs free energy for the Kerr black hole. Figure 12 illustrates the Gibbs free energy influenced by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The values of q are determined based on the stability of the black hole as indicated in equation (43).

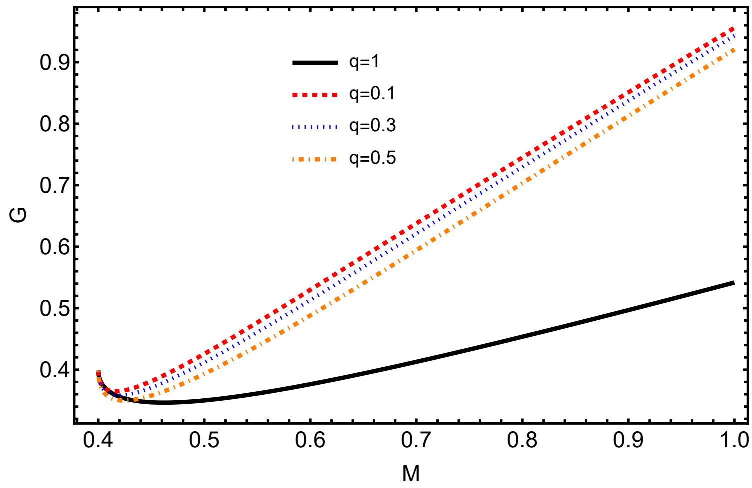

Figure 12: The illustration of the Gibbs free energy inspired by the nonextensive Tsallis entropy for the Kerr black hole for various values of q versus M with $a = 0.4$. The $q = 1$ case is for the ordinary Gibbs free energy of the Kerr black hole.

4.2 Nonextensive Kaniadakis Entropy

4.2.1 Schwarzschild Black Hole

We aim to affect the nonextensive Kaniadakis entropy on the Schwarzschild black hole. Thus, we apply equation (4) to equation (32) to the form

$$S_K = \frac{\sinh\left(4\pi K M^2\right)}{K}\,, \tag{45}$$

which is the Kaniadakis entropy of the Schwarzschild black hole. The graph in Figure 13 presents the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The values of K in this figure are arbitrarily chosen. From Figure 13, it is evident that increasing the value of K leads to an increase in the Kaniadakis entropy of the Schwarzschild black hole, and these values are larger than the values of the ordinary Bekenstein entropy of the Schwarzschild black hole.

Equations (45) and (5) can give us the Hawking temperature of the Schwarzschild black hole in the nonextensive Kaniadakis entropy setup as follows

$$T_H = \frac{\operatorname{sech}\left(4\pi K M^2\right)}{8\pi M}\,. \tag{46}$$

For $K \to 0$, the ordinary Hawking temperature for the Schwarzschild black hole will recover from equation (46). Comparing equation (46) with the relation $T_H = \kappa/2\pi$, we see that nonextensive Kaniadakis statistics even affect on the gravitational theory outcomes such that surface gravity becomes

$$\kappa = \frac{\operatorname{sech}\left(4\pi K M^2\right)}{4M}\,,$$

which depends on q parameter. The behavior of the Hawking temperature influenced by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M is depicted in Figure 14. The values of K in this figure are arbitrarily chosen. From

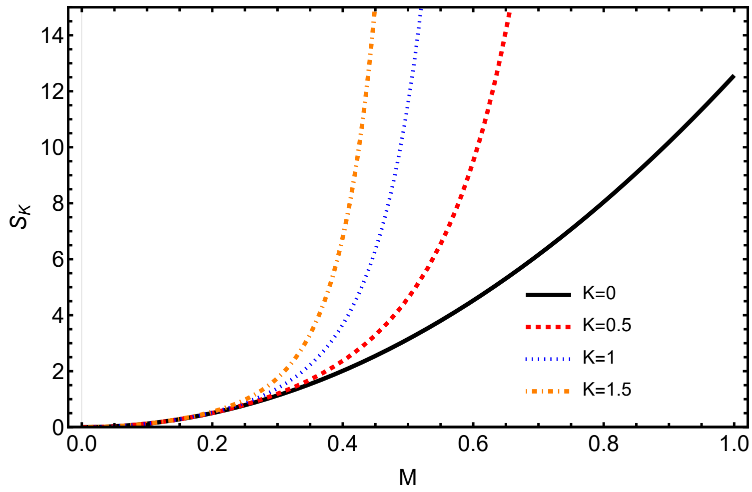

Figure 13: The illustration of the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The $K = 0$ case is for the ordinary Bekenstein entropy of the Schwarzschild black hole.

Figure 14, it is evident that increasing the value of K results in a reduction of the Hawking temperature influenced by the nonextensive Kaniadakis entropy for the Schwarzschild black hole.

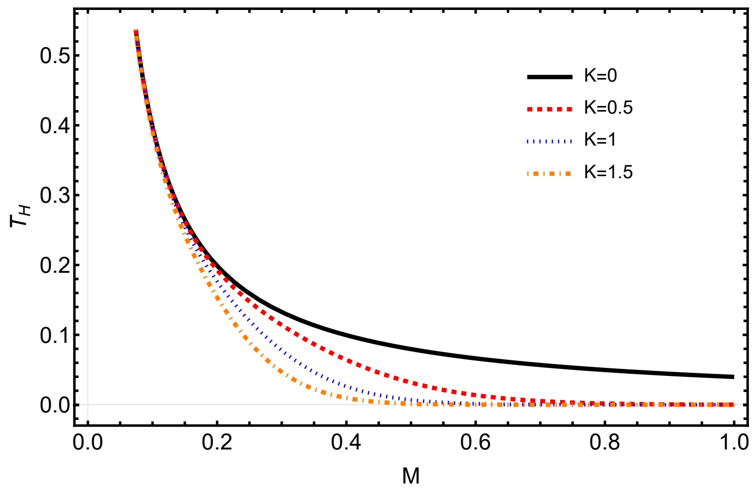

Figure 14: The illustration of the Hawking temperature inspired by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The $K = 0$ case is for the ordinary Hawking temperature of the Schwarzschild black hole.

We now employ equations (6) and (45) to find the heat capacity inspired by nonextensive Kaniadakis entropy as follows

$$C = -\frac{8\pi M^2 \cosh^2\left(4\pi K M^2\right)}{8\pi K M^2 \sinh\left(4\pi K M^2\right) + \cosh\left(4\pi K M^2\right)}\,. \tag{47}$$

In the condition of $K \to 0$, the ordinary heat capacity for the Schwarzschild black hole recovers from equation (47). One can verify that the heat capacity inspired by nonextensive Kaniadakis entropy (45) is negative. Hence, the Schwarzschild black hole in the nonextensive Kaniadakis statistics is thermodynamically unstable. Figure 15 illustrates the heat capacity influenced by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The values of K in this figure are arbitrarily chosen. From this figure, it is observed that the Schwarzschild black hole is unstable with both the nonextensive Kaniadakis entropy and the ordinary Bekenstein entropy.

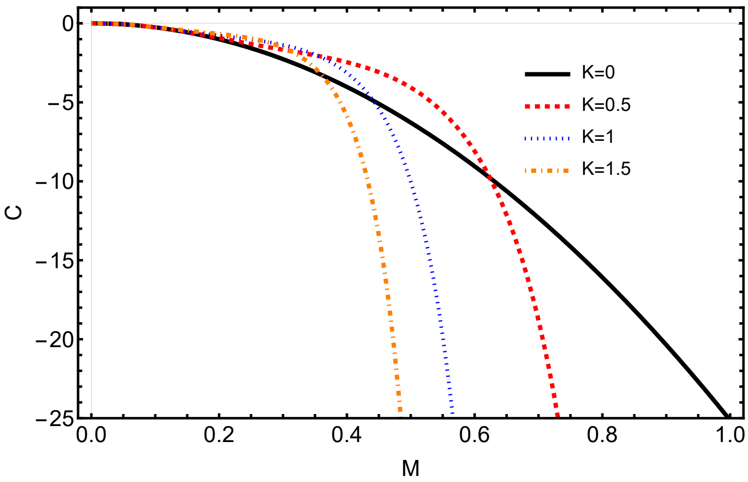

Figure 15: The illustration of the heat capacity inspired by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The $K = 0$ case is for the ordinary heat capacity of the Schwarzschild black hole.

From equation (8) and equations (45) and (46), one can find the Gibbs free energy for the Schwarzschild black hole affected by nonextensive Kaniadakis entropy as follows

$$G = M - \frac{\tanh\left(4\pi K M^2\right)}{8\pi K M}\,, \tag{48}$$

which again in the limit of $K \to 0$, reduces to the ordinary Gibbs free energy for the Schwarzschild black hole. Figure 16 depicts the Gibbs free energy influenced by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The values of K in this figure are arbitrarily chosen. It is observed from Figure 16 that increasing the value of K leads to an increase in the Gibbs free energy influenced by the nonextensive Kaniadakis entropy for the Schwarzschild black hole.

4.2.2 Reissner-Nordström Black Hole

We are going to explore the effects of the nonextensive Kaniadakis entropy on the Reissner-Nordström black hole. Using equations (13) and (32), we obtain the following expression

$$S_K = \frac{\sinh\left(\pi K\left(M + \sqrt{M^2 - Q^2}\right)^2\right)}{K}\,. \tag{49}$$

This is the Kaniadakis entropy of the Reissner-Nordström black hole. The graph in Figure 17 illustrates the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole

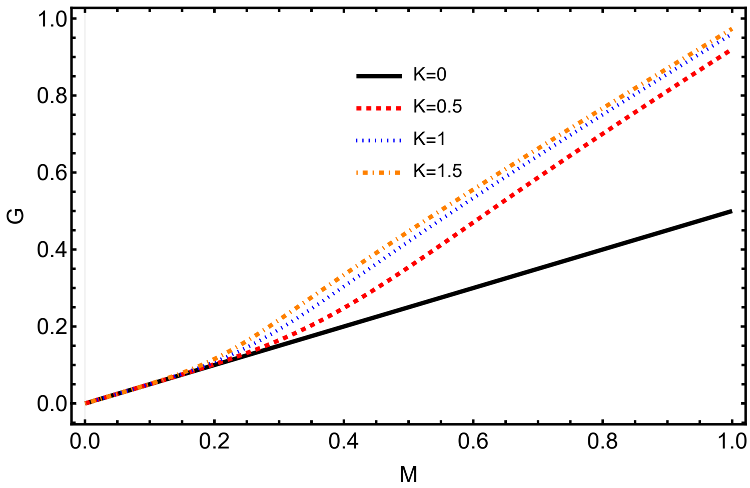

Figure 16: The illustration of the Gibbs free energy inspired by the nonextensive Kaniadakis entropy for the Schwarzschild black hole for various values of K versus M. The $K = 0$ case is for the ordinary Gibbs free energy of the Schwarzschild black hole.

for various values of K versus M with $Q = 1$. The values of K in this figure are arbitrarily chosen. Increasing the value of K leads to an increase in the Kaniadakis entropy of the Reissner-Nordström black hole, as observed in Figure 17, whereas the ordinary Bekenstein entropy of the Reissner-Nordström black hole associated with $K = 0$ is smaller than the Kaniadakis entropy.

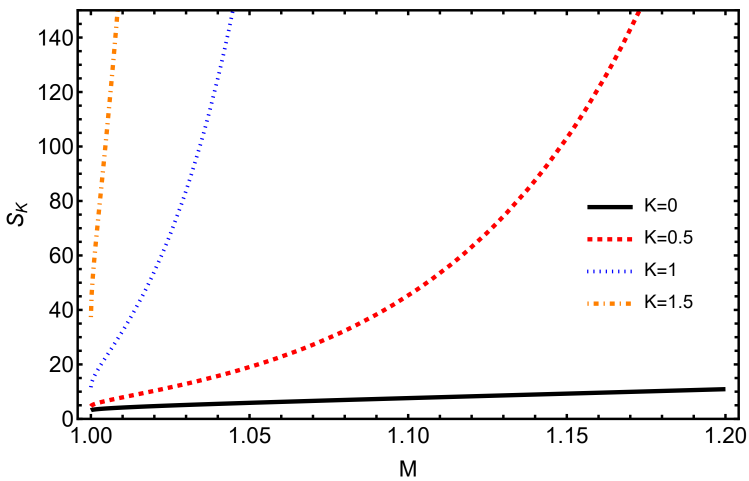

Figure 17: The illustration of the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The $K = 0$ case is for the ordinary Bekenstein entropy of the Reissner-Nordström black hole.

Equations (49) and (5) can give us the Hawking temperature of the Reissner-Nordström black hole in the nonextensive Kaniadakis entropy formalism as

$$T_H = \frac{\sqrt{M^2 - Q^2}\,\mathrm{sech}\left(\pi K \left(M + \sqrt{M^2 - Q^2}\right)^2\right)}{2\pi \left(M + \sqrt{M^2 - Q^2}\right)^2}. \tag{50}$$

For $K \to 0$, the ordinary Hawking temperature for the Reissner-Nordström black hole will recover from equation (50). The graph in Figure 18 illustrates the Hawking temperature influenced by the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The values of K in this figure are arbitrarily chosen. It can be observed from Figure 18 that the effect of the parameter K of the nonextensive Kaniadakis entropy on the Hawking temperature is to reduce its value. The different curves associated with various K values are not very distinctive.

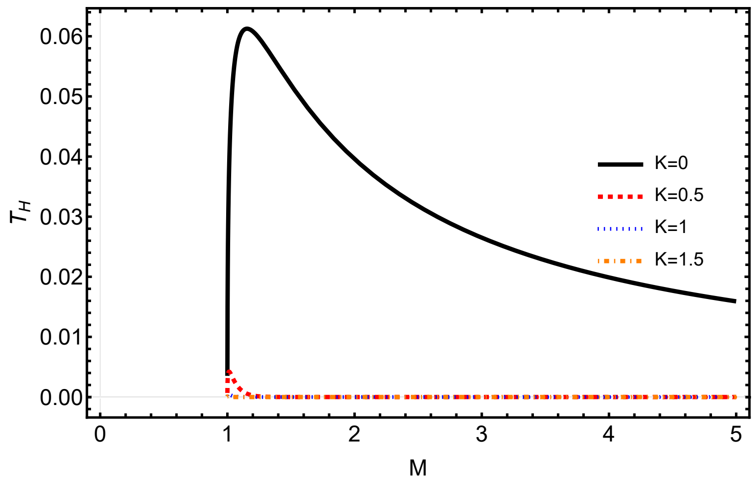

Figure 18: The illustration of the Hawking temperature inspired by the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The $K = 0$ case is for the ordinary Hawking temperature of the Reissner-Nordström black hole.

We now employ equations (6) and (49) to find the heat capacity for the Reissner-Nordström black hole inspired by nonextensive Kaniadakis entropy as follows

$$C = \frac{-2\pi\sqrt{M^2 - Q^2}\left(M + \sqrt{M^2 - Q^2}\right)^2 \cosh^2\left(\pi K \left(M + \sqrt{M^2 - Q^2}\right)^2\right)}{2\pi K X \sinh\left(\pi K \left(M + \sqrt{M^2 - Q^2}\right)^2\right) + \left(2\sqrt{M^2 - Q^2} - M\right)\cosh\left(\pi K \left(M + \sqrt{M^2 - Q^2}\right)^2\right)}, \tag{51}$$

where we have defined

$$X = 2M^3 + 2M^2\sqrt{M^2 - Q^2} - Q^2\sqrt{M^2 - Q^2} - 2MQ^2.$$

In the condition of $K \to 0$, the ordinary heat capacity for the Reissner-Nordström black hole recovers from equation (47). One can verify that the heat capacity inspired by nonextensive Kaniadakis entropy (45) is negative. Hence, the Reissner-Nordström black hole in the nonextensive Kaniadakis statistics is thermodynamically unstable. The graph in Figure 19 illustrates the heat capacity influenced by the nonextensive Kaniadakis entropy for the

Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The values of K in this figure are arbitrarily chosen. From this figure, it is evident that both the heat capacity influenced by the nonextensive Kaniadakis entropy and the ordinary heat capacity of the Reissner-Nordström black hole are negative, resulting in the instability of the black hole.

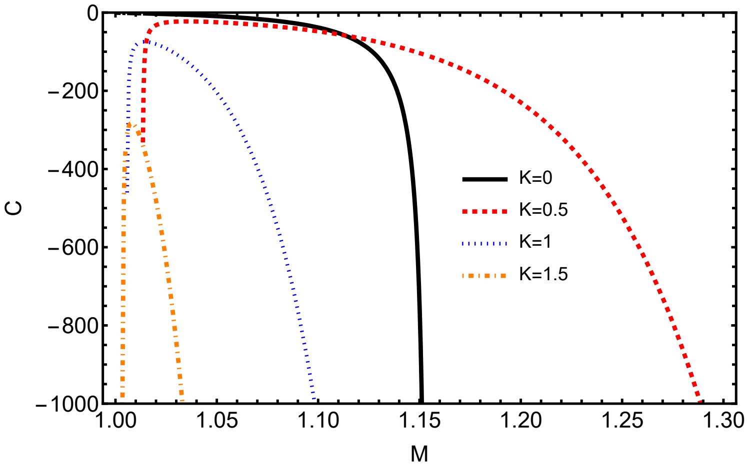

Figure 19: The illustration of the heat capacity inspired by the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The $K = 0$ case is for the ordinary heat capacity of the Reissner-Nordström black hole.

From equation (8) and equations (49) and (50), one can find the Gibbs free energy for the Reissner-Nordström black hole affected by nonextensive Kaniadakis entropy as follows

$$G = M - \frac{\sqrt{M^2 - Q^2}\tanh\left(\pi K\left(M + \sqrt{M^2 - Q^2}\right)^2\right)}{2\pi K\left(M + \sqrt{M^2 - Q^2}\right)^2}\,, \tag{52}$$

which again in the limit of $K \to 0$, reduces to the ordinary Gibbs free energy for the Reissner-Nordström black hole. Figure 20 is the illustration of the Gibbs free energy inspired by the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The values of K in this figure are arbitrarily chosen. The values of the Gibbs free energy inspired by the nonextensive Kaniadakis entropy are larger than the ordinary Gibbs free energy of the black hole as seen in Figure 20.

4.2.3 Kerr Black Hole

Exploring the effects of the nonextensive Kaniadakis entropy on the Kerr black hole is our plan here. We utilize equations (22) and (32) to find the following result

$$S_K = \frac{\sinh\left(\pi K\left(\left(M + \sqrt{M^2 - a^2}\right)^2 + a^2\right)\right)}{K}\,. \tag{53}$$

This is the Kaniadakis entropy of the Kerr black hole. Figure 21 is the illustration of the nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M

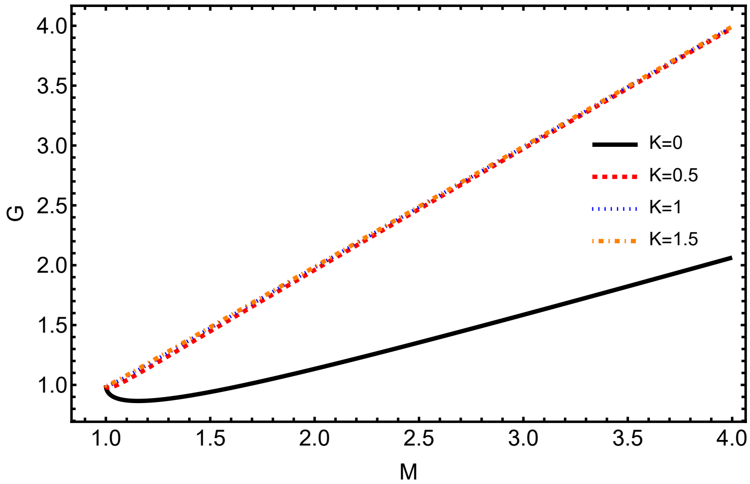

Figure 20: The illustration of the Gibbs free energy inspired by the nonextensive Kaniadakis entropy for the Reissner-Nordström black hole for various values of K versus M with $Q = 1$. The $K = 0$ case is for the ordinary Gibbs free energy of the Reissner-Nordström black hole.

with $a = 0.4$. The values of K in this figure are arbitrarily chosen. From Figure 21 we see that increasing the values of K leads to amplifying the Kaniadakis entropy of the Kerr black hole.

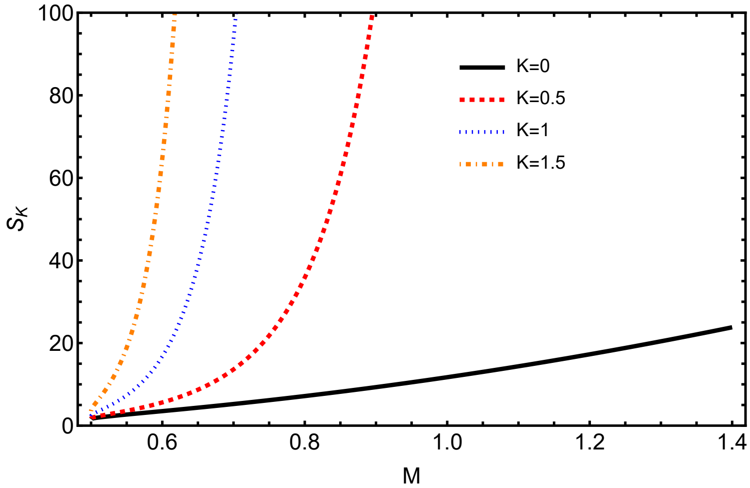

Figure 21: The illustration of the nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The $K = 0$ case is for the ordinary Bekenstein entropy of the Kerr black hole.

Equations (53) and (5) can give us the Hawking temperature of the Kerr black hole in the nonextensive Kaniadakis entropy formalism as

$$T_H = \frac{\sqrt{M^2 - a^2}\,\text{sech}\left(2\pi KM\left(M + \sqrt{M^2 - a^2}\right)\right)}{2\pi\left(M + \sqrt{M^2 - a^2}\right)^2} \,. \tag{54}$$

For $K \to 0$, the ordinary Hawking temperature for the Kerr black hole will recover from equation (54). Figure 22 shows the illustration of the Hawking temperature inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The values of K in this figure are arbitrarily chosen. From Figure 22, we find that increasing K leads to decrease the Hawking temperature inspired by nonextensive Kaniadakis entropy for the Kerr black hole.

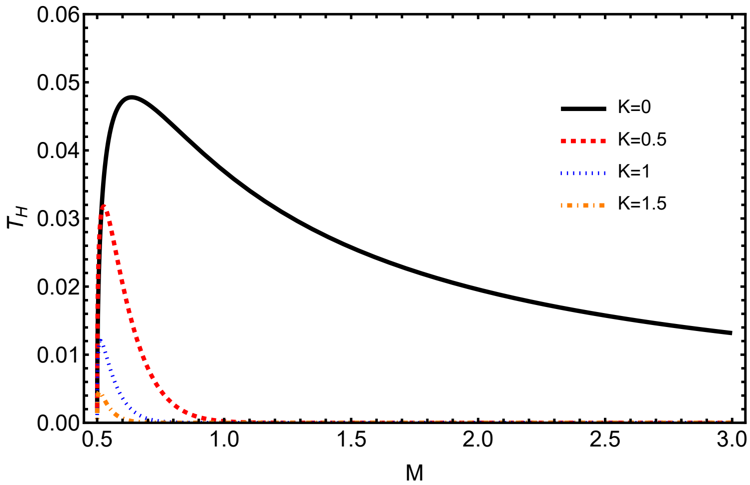

Figure 22: The illustration of the Hawking temperature inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The $K = 0$ case is for the ordinary Hawking temperature of the Kerr black hole.

We now employ equations (6) and (53) to find the heat capacity for the Kerr black hole inspired by nonextensive Kaniadakis entropy as

$$C = -\frac{2\pi\sqrt{M^2-a^2}\left(M+\sqrt{M^2-a^2}\right)^2\cosh^2\left(2\pi KM\left(M+\sqrt{M^2-a^2}\right)\right)}{2\pi KY\sinh\left(2\pi KM\left(M+\sqrt{M^2-a^2}\right)\right)-\left(M-2\sqrt{M^2-a^2}\right)\cosh\left(2\pi KM\left(M+\sqrt{M^2-a^2}\right)\right)}, \tag{55}$$

where we have defined

$$Y = 2M^2\left(M+\sqrt{M^2-a^2}\right) - a^2\left(\sqrt{M^2-a^2}+2M\right).$$

In the condition of $K \to 0$, the ordinary heat capacity for the Kerr black hole recovers from equation (55). One can verify that the heat capacity inspired by nonextensive Kaniadakis entropy (55) is negative. Hence, the Kerr black hole in the nonextensive Kaniadakis statistics is thermodynamically unstable. Figure 23 shows the illustration of the heat capacity inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The values of K in this figure are arbitrarily chosen. From Figure 23, we find that both heat capacity inspired by nonextensive Kaniadakis entropy and the ordinary heat capacity for the Kerr black hole are negative and consequently the Kerr black hole is unstable in this parameter space chosen by us. Also, increasing K leads to decrease the Hawking temperature inspired by nonextensive Kaniadakis entropy for the Kerr black hole.

From equation (8) and equations (53) and (54), one can find the Gibbs free energy for

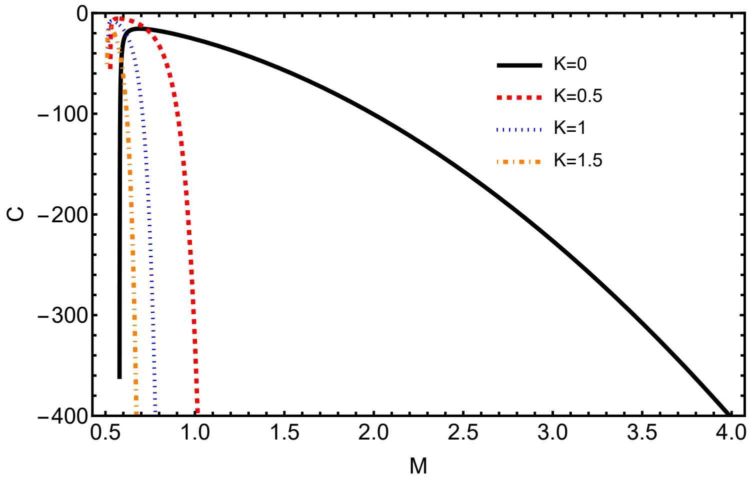

Figure 23: The illustration of the heat capacity inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The $K = 0$ case is for the ordinary heat capacity of the Kerr black hole.

the Kerr black hole affected by nonextensive Kaniadakis entropy as follows

$$G = M - \frac{\sqrt{M^2 - a^2}\tanh\left(2\pi KM\left(M + \sqrt{M^2 - a^2}\right)\right)}{2\pi K\left(M + \sqrt{M^2 - a^2}\right)^2}. \tag{56}$$

Again in the limit of $K \to 0$, reduces to the ordinary Gibbs free energy for the Kerr black hole. Figure 24 shows the illustration of the Gibbs free energy inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The values of K in this figure are arbitrarily chosen. From Figure 24, we find that the Gibbs free energy inspired by nonextensive Kaniadakis entropy is bigger then the ordinary one for the Kerr black hole.

5 Conclusions

In this work, we first provided an overview of different black hole metric backgrounds, including Schwarzschild, Reissner-Nordström, and Kerr black holes. We then investigated the thermodynamic properties of these black hole systems from the perspective of extensive entropy. Additionally, we studied the stability conditions for these black holes. Nonextensive entropy formulations can result in thermodynamic properties that differ from those of the standard Bekenstein-Hawking entropy. In this context, we reviewed two nonextensive entropies: Tsallis and Kaniadakis, along with some of their thermodynamic properties. As the primary purpose of this paper, we applied the nonextensive entropies–specifically, Tsallis and Kaniadakis entropies to the thermodynamic analysis of Schwarzschild, Reissner-Nordström, and Kerr black holes, all within the framework of general relativity. This is crucial since the strong gravitational fields of black holes can be studied through extensive thermodynamics, which eliminates long-term and strong regimes. We also explored the stability conditions for the aforementioned black holes in the context of nonextensive Tsallis and Kaniadakis entropies. Our findings indicated that the Schwarzschild black hole, when influenced by

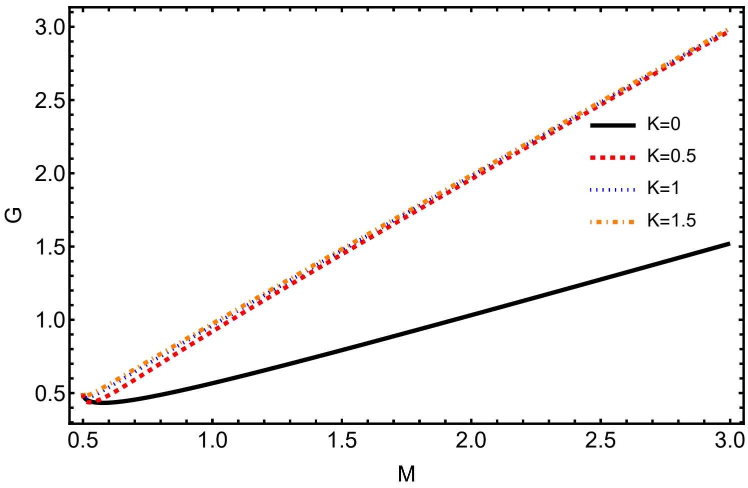

Figure 24: The illustration of the Gibbs free energy inspired by nonextensive Kaniadakis entropy for the Kerr black hole for various values of K versus M with $a = 0.4$. The $K = 0$ case is for the ordinary Gibbs free energy of the Kerr black hole.

nonextensive Tsallis entropy, can exhibit stable behavior under certain parameter conditions. This stability result was also observed for the Reissner-Nordström and Kerr black holes. However, in the case of nonextensive Kaniadakis entropy, we discovered that there are no stable conditions for these black hole configurations. It is worth mentioning that a similar study has investigated the nonextensive entropies and related thermodynamic quantities in Renyi, Tsallis, Sharma-Mittal, Kaniadakis, and Barrow nonextensive statistics, all within the context of the generalized uncertainty principle (GUP) for spherically symmetric Schwarzschild black holes [47]. The findings indicate that, for the Renyi, Tsallis, and Sharma-Mittal entropies, black holes can exhibit stability under certain conditions in both GUP and non-GUP scenarios. However, in the case of Kaniadakis entropy, black holes are not stable at all, which is consistent with our results.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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