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Null Geodesics of a Symmergent Black Hole

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Abstract. In this paper, null geodesics for a symmergent black hole are investigated. The geodesic equation of this space time is solved analytically according to the Weierstrass elliptic function. Also, the effective potential is obtained and plotted. Finally, using the form of effective potential and obtained analytical solution of geodesic equations, some possible types of orbits related to null geodesics are demonstrated.

Keywords: Black Hole; Null Geodesics; Elliptic Function; Analytical Solution.

1 Introdution

Analytical solutions of geodesic equations help to understand the gravitational effects of black holes. These analytical solutions can be effective for checking various experimental predictions such as perihelion shift, Lens-Teering effect, light deviation and gravitational time delay. Due to the importance of the subject, so far many researchers have tried to analyze the geodetic equations of many different space times with the help of analytical solutions [1–12]. Symmergent gravity is a special case of generalized $f(R)$ gravity and can be said to be gravity with quadratic curvature and a finite cosmological constant. It can be said that the solution of symmergent black hole was first investigated by Cimdiker and colleagues in [13], and different physical characteristics of this type of black hole, including weak lensing, shadow radius, quasi-periodic, oscillations, and other information about this theory, have been analyzed by various researchers [14–21]. In addition to the previous works regarding symmergent black hole, it can be important to study the motion of light rays around this black hole. Certainly, investigating the state of light rays motion requires investigating geodesic equations and solving them analytically. Here, Weirstrass elliptic function is used for the analytical solution of null geodesic equations. The structure of this paper is as follows: In section (2), a brief overview of a symmergent black hole space time is given. Analytical solution of null geodesic equations, effective potential and lightlike orbits are studied in section (3). Finally, the conclusion is presented in section (4).

2 symmergent black hole

In this section, the metric of a symmergent black hole is studied. The action used in this theory can be written as follows [20,21]

$$\mathcal{I} = \int d^4x\sqrt{-g}\left[\frac{\mathcal{R}}{16\pi G} - \frac{c_0}{16}\mathcal{R}^2 - V_0 - \mathcal{L}_m\right], \tag{1}$$

where $\mathcal{L}_m$ is the matter Lagrangian, $\mathcal{R}$ is the Ricci scalar, c_0 is the quadratic curvature coefficient (the symmergent parameter), and V_0 is the vacuum energy density. Considering the above action and the corresponding field equations, the metric of a static spherically symmergent black hole spacetime can be written as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2, \tag{2}$$

and

$$f(r) = 1 - \frac{2MG}{r} - \frac{(1-\alpha)}{24\pi c_0 G}r^2, \tag{3}$$

where as mentioned earlier, c_0 is the symmergent parameter (loop coefficient) and also α is an integration constant. For the case of $\alpha = 1$ or $c_0 = \infty$, the metric function (3) converts to the Schwarzschild black hole metric. More details of this solution can be found in Refs. [20,21].

3 Analytical solution of geodesic equations

In this section, the geodesic equations and analytical solutions are studied. The geodesic motion is described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0, \tag{4}$$

where $\Gamma^{\mu}_{\rho\sigma}$ is the Christoffel symbol. The conserved energy and angular momentum as a constant of motions are obtained by the normalization condition $\frac{1}{2}g_{\mu\nu}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} = \frac{1}{2}\epsilon$, where for massive particles $\epsilon = 1$ and for light $\epsilon = 0$,

$$E = g_{tt}\frac{dt}{ds}, \tag{5}$$

$$L = g_{\varphi\varphi}\frac{d\varphi}{ds} = r^2\frac{d\varphi}{ds}. \tag{6}$$

So, the geodesic equations can be obtained as

$$\left(\frac{dr}{ds}\right)^2 = E^2 - \left(1 - \frac{2MG}{r} - \frac{(1-\alpha)}{24\pi c_0 G}r^2\right)\left(\varepsilon + \frac{L^2}{r^2}\right), \tag{7}$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{L^2}\left(E^2 - (1 - \frac{2MG}{r} - \frac{(1-\alpha)}{24\pi c_0 G}r^2)\right)\left(\varepsilon + \frac{L^2}{r^2}\right). \tag{8}$$

Equations (7) and (8) give a description of the dynamics of the geodesic motion. The effective potential can be obtained from equation (7) as

$$V_{eff} = \left(1 - \frac{2MG}{r} - \frac{(1-\alpha)}{24\pi c_0 G}r^2\right)\left(\varepsilon + \frac{L^2}{r^2}\right). \tag{9}$$

Plot of effective potential is shown in Figure (1). With $G = 0$ and the dimensionless quantities

$$\begin{aligned} \tilde{r} &= \frac{r}{M}, \\ \tilde{\mathcal{L}} &= \frac{M^2}{L^2}, \\ \tilde{c} &= \frac{c_0}{M}, \end{aligned} \tag{10}$$

equation (8) can be written as

$$\left(\frac{d\tilde{r}}{d\varphi}\right)^2 = \frac{\tilde{\mathcal{L}}\varepsilon\,(1-\alpha)\,\tilde{r}^6}{24\tilde{c}\pi} + ((1-\alpha) + (E^2 - \varepsilon)\tilde{\mathcal{L}})\,\tilde{r}^4 + 2L\varepsilon\tilde{r}^3 - \tilde{r}^2 + 2\tilde{r}. \tag{11}$$

The necessary condition for the existence of a geodesic is $R(\tilde{r}) \geq 0$. Thus, the zeros of $R(\tilde{r})$, determine the type of geodesic.

3.1 Null geodesics

For $\varepsilon = 0$ and $u = \frac{1}{\tilde{r}}$, equation (11) is converted into the following

$$\left(\frac{du}{d\varphi}\right)^2 = 2u^3 - u^2 + ((1-\alpha) + E^2\tilde{\mathcal{L}}) = P_3(u). \tag{12}$$

Equation (12) is of elliptic type and with the substitution $u = 2y + \frac{1}{6}$, can be transformed to the Weierstrass form

$$\left(\frac{dy}{d\varphi}\right)^2 = 4y^3 - g_2 y - g_3 = P_3(y), \tag{13}$$

$$g_2 = \frac{1}{12},$$
$$g_3 = \frac{1}{216} - \frac{1}{4}((1-\alpha) + E^2\tilde{\mathcal{L}}).$$

The analytical solution of equation (13) is given by

$$y(\varphi) = \wp(\varphi - \varphi_{in}), \tag{14}$$

and so,

$$\tilde{r}(\varphi) = \frac{1}{2\wp(\varphi - \varphi_{in}; g_2, g_3) + \frac{1}{6}}, \tag{15}$$

in which

$$\varphi_{in} = \varphi_0 + \int_{y_0}^{\infty} \frac{dz}{\sqrt{4y^3 - g_2 - g_3}}, \tag{16}$$

and

$$y_0 = \frac{1}{2\tilde{r_0}} - \frac{1}{12},$$

depends only on the initial values φ_0 and $\tilde{r_0}$.

3.2 Orbits

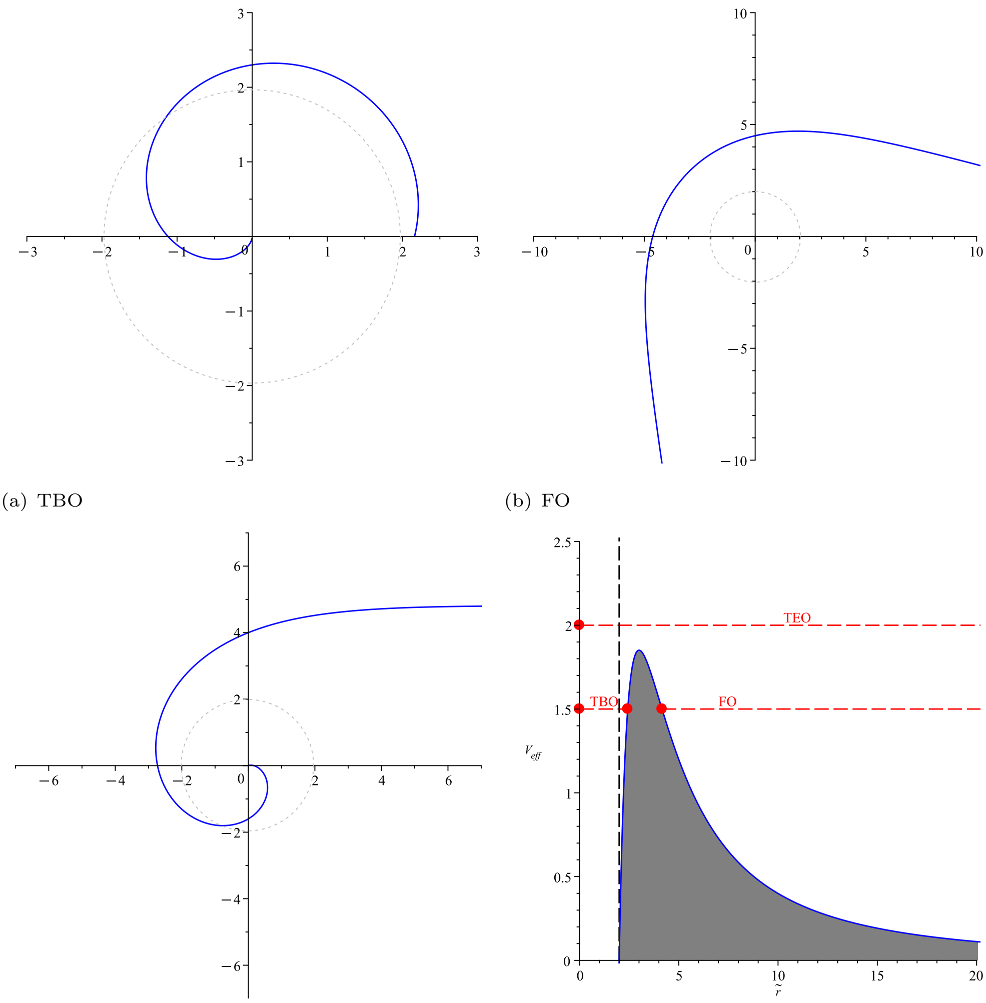
Here, using the figure of the effective potential and obtained analytical solution, some of possible orbits are demonstrated in Figure 1. Here, different types of orbits can be identified as

1. Terminating bound orbit (TBO): r starts in $(0, r_a]$ for $0 < r_a < \infty$ and falls into the singularity at $r = 0$.
2. Flyby orbit (FO): r starts from ∞, then approaches a periapsis $r = r_p$ and goes back to ∞.
3. Terminating escape orbit (TEO): r comes from ∞ and falls into the singularity at $r = 0$.

Examples of such orbits along with the shape of the effective potential are demonstrated in Figure 1.

4 Conclusion

In this paper, null geodesics for a symmergent black hole were investigated. The equations of geodesic motions were obtained and solved according to Weierstrass elliptic function. Moreover, by the help of the obtained analytical solution and the effective potential, we have shown various possible orbits for light rays such as TBO, FO and TEO. These results can be useful information for orbits around heavy objects, including the light deflection.



(a) TBO (b) FO (c) TEO (d)

Figure 1: (a, b, c): Examples of different types of orbits. The blue curves indicate the orbits and the black dashed circle depicts the position of the horizon. (d): Plot of the effective potential together with examples of energies for the different types of orbits. The red dashed lines correspond to energies. The red dots mark the zeros of the polynomial R, which are the turning points of the orbits. The vertical dashed line represents the horizon of the black hole.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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