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**Regular article**

# Thermodynamic Properties of Strongly Gravitating Systems

Mir Hameeda[1] · Mario C. Rocca[2]

[1] Department of Physics, S.P. Collage, Srinagar, Kashmir, 190001 India;
Inter University Centre for Astronomy and Astrophysics , Pune India;
E-mail: hme123eda@gmail.com

[2] Departamento de Física, Facultad de Ciencias Exactas Universidad Nacional de La Plata, Argentina;
Departamento de Matemáticas, Facultad de Ciencias Exactas Universidad Nacional de La Plata, Argentina;
Consejo Nacional de Investigaciones Científicas y Tecnológicas (IFLP-CCT-CONICET)-C. C. 727, 1900 La Plata, Argentina;
Corresponding Author E-mail: mariocarlosrocca@gmail.com

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**Abstract**. This paper investigates the thermodynamic properties of strongly interacting gravitational systems. By deriving the partition function for such systems, the authors obtain expressions for various thermodynamic quantities, including internal energy, specific heat, Helmholtz free energy, entropy, chemical potential, and pressure. The paper introduces an interaction parameter that quantifies the degree of non-ideality in the system, and explores its effects on the thermodynamic properties. The authors also calculate the moments generating function, which provides information about the distribution of particle positions. Additionally, the distribution function for the system is derived. The paper highlights the existence of an upper bound temperature beyond which the partition function becomes negative, indicating a limit on the validity of the model.

*Keywords*: Thermodynamics; Holography; Gravitation.

# 1 Introduction

The vastness of intergalactic distances far exceeds the typical length scale of individual galaxies by many orders of magnitude. Consequently, a system of galaxies can be effectively modeled as a collection of point particles. This simplification allows for the application of standard methods from statistical mechanics to analyze the system [1]. Furthermore, this approach facilitates the exploration of the thermodynamic limits of such an interacting system of galaxies, enabling the derivation of relevant thermodynamic quantities. By treating galaxies as point particles, we can apply various statistical mechanics frameworks to gain insights into the macroscopic behavior of the system. This includes calculating quantities such as entropy, temperature, and free energy, which describe the overall state of the galaxy system. Additionally, understanding the interactions between these point particles helps in elucidating the large-scale structure and dynamics of the universe. Analyzing the thermodynamic properties of a galaxy system provides valuable information about its equilibrium states and the conditions under which phase transitions might occur. These insights are crucial for developing a comprehensive understanding of cosmic evolution and the underlying physical principles governing the universe. As such, the study of intergalactic systems through the lens of statistical mechanics and thermodynamics is a powerful tool in the field of cosmology, offering profound implications for our knowledge of the universe's fundamental nature [2]. The distribution of galaxies is profoundly influenced by gravitational forces, as highlighted by Ref. [3]. Gravitational force is a fundamental factor not only in the clustering of galaxies but also in the formation of their large-scale structures. Understanding the characterization of galactic clusters on a vast scale is essential for comprehending the evolution and distribution of galaxies throughout the Universe [4]. One of the standard methods to study the formation and evolution of the Universe involves analyzing correlation functions through observational data and N-body computer simulations [5]. In gravitating systems where interactions occur in pairs, these correlation functions are critical as they determine the thermodynamical properties, including gravitational effects. The importance of these studies lies in their ability to provide insights into the underlying physical processes governing galactic dynamics and interactions. By examining the spatial distribution and movement of galaxies, researchers can infer the influence of dark matter and dark energy, as well as gain a deeper understanding of cosmic expansion and the overall architecture of the cosmos. Furthermore, advancements in computational techniques and observational technologies have significantly enhanced our ability to model and simulate complex gravitational interactions in the Universe. This progress has allowed for more precise predictions and a better grasp of how galaxies coalesce into larger structures such as galaxy clusters and superclusters.

The study of galactic clustering, driven by the gravitational interactions between galaxies, has become a topic of profound interest in the field of astrophysics. These gravitational interactions play a critical role in shaping the large-scale structure of the universe, influencing the formation and evolution of galaxies. By examining how galaxies group together in clusters, researchers can gain valuable insights into the underlying principles governing cosmic structure and the distribution of dark matter. This area of research not only helps in understanding the dynamics and history of galaxy formation but also provides essential clues about the overall composition and fate of the universe. The complexity of these interactions necessitates sophisticated observational techniques and theoretical models, making galactic clustering a vibrant and evolving area of scientific inquiry [6–8].

Strongly gravitating systems, such as dense stellar clusters, galactic nuclei, and certain astrophysical objects, exhibit complex interactions that cannot be adequately described by simple models of ideal gases. In these systems, gravitational forces dominate over other interactions, leading to highly non-linear and correlated dynamics. Understanding the thermodynamic properties of strongly gravitating systems is crucial for gaining insights into their formation, evolution, and observable characteristics [9]. Previous studies have primarily focused on dilute systems or perturbative approaches, where the gravitational interactions are treated as small perturbations. However, in strongly gravitating systems, the interactions are so strong that non-perturbative methods are necessary to capture the full complexity of the problem [10]. One approach to studying such systems is through the formalism of statistical mechanics, which provides a framework for describing the collective behavior of many-body systems. By deriving the partition function, which encodes the statistical properties of the system, one can obtain expressions for various thermodynamic quantities, such as internal energy, specific heat, free energy, and entropy [11]. Early attempts to derive the partition function for gravitational systems faced challenges due to the long-range nature of the gravitational force and the intrinsic divergences that arise in the calculations. However, recent advances in techniques such as dimensional regularization have provided new avenues for addressing these issues. This paper aims to develop a theoretical framework for studying the thermodynamics of strongly interacting gravitational systems by deriving the partition function and exploring its consequences. Specifically, we employ dimensional regularization techniques to obtain a well-defined partition function, from which we derive expressions for various thermodynamic quantities, including internal energy, specific heat, Helmholtz free energy, entropy, chemical potential, and pressure. To account for the non-ideality of the system, we introduce an interaction parameter that quantifies the degree of deviation from ideal gas behavior. This parameter allows us to study the effects of strong interactions on the thermodynamic properties and explore the transition from dilute to strongly interacting regimes. Furthermore, we calculate the moments generating function, which provides information about the distribution of particle positions within the system. This analysis can shed light on the spatial correlations and clustering properties that arise due to the strong gravitational interactions. The understanding of strongly gravitating systems has implications not only in astrophysics but also in the broader context of gravitational physics and the holographic principles that govern the relationship between gravity and quantum mechanics.

The study of strongly gravitating systems and their thermodynamic properties has potential connections to holographic principles in theoretical physics. Holography, as proposed by the holographic principle, suggests that the information contained within a volume of space can be fully described by the information encoded on its boundary. This principle has profound implications for our understanding of gravity and quantum mechanics, particularly in the context of black holes and the AdS/CFT correspondence (Anti-de Sitter/Conformal Field Theory correspondence). Strongly gravitating systems, such as dense stellar clusters and certain astrophysical objects, can exhibit gravitational effects that may be relevant to the holographic description of gravity. The thermodynamic properties of these systems, as studied in this paper, could potentially provide insights into the holographic nature of gravity and the relationship between bulk and boundary descriptions.

## 2 Partition function of a strongly interacting system

The partition function $\mathcal{Z}$ in $\nu$ dimensions is given by

$$\mathcal{Z}_\nu = -\frac{1}{N!}\int\limits_{-\infty}^{\infty} d^\nu x \int\limits_{-\infty}^{\infty} d^\nu p \exp\left[\beta\left(\frac{N(N-1)Gm^2}{2r} - \frac{Np^2}{2m}\right)\right], \tag{1}$$

which can be written as

$$\mathcal{Z}_\nu = -\frac{1}{N!}\left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \int_0^\infty r^{\nu-1}dr \int_0^\infty p^{\nu-1}dp \exp\left(\beta\left(\frac{N(N-1)Gm^2}{2r}\right)\exp\left(-\frac{Np^2}{2m}\right)\right). \quad (2)$$

It can be reduced to

$$\mathcal{Z}_\nu = -\frac{1}{N!}\left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \int_0^\infty r^{\nu-1}dr \exp\left(\beta\left(\frac{N(N-1)Gm^2}{2r}\right)\int_0^\infty p^{\nu-1}dp \exp\left(-\frac{Np^2}{2m}\right)\right). \quad (3)$$

Using integral

$$\int_0^\infty r^{\nu-1}dr \exp\left(\beta\left(\frac{N(N-1)Gm^2}{2r}\right)\right) = \cos(\pi\nu)\left(\frac{N(N-1)\beta Gm^2}{2}\right)^\nu \Gamma(-\nu), \quad (4)$$

and

$$\int_0^\infty p^{\nu-1}dr \exp\left(-\beta\left(\frac{Np^2}{2m}\right)\right) = \frac{2^{-\nu}\left(\frac{N\beta}{2m}\right)^{\frac{-\nu}{2}}\sqrt{\pi}\Gamma(\nu)}{\Gamma\left(\frac{\nu+1}{2}\right)}, \quad (5)$$

the partition function $\mathcal{Z}$ is obtained as

$$\mathcal{Z} = -\frac{1}{N!}4\pi\cos(\pi\nu)\left(\frac{N(N-1)^2\beta G^2m^5\pi^2}{4}\right)^{\frac{\nu}{2}} \frac{\Gamma(\nu)\Gamma(-\nu)}{\Gamma\left(\frac{\nu}{2}\right)^2\Gamma\left(\frac{\nu+1}{2}\right)}. \quad (6)$$

This diverges at $\nu = 3$, thus we appeal to the dimensional regularization (DR) and write the partition function independent of the terms containing $\nu - 3$. Thus we get

$$\mathcal{Z} = -\frac{1}{N!}\frac{1}{3\sqrt{\pi}}\left(\frac{\pi^2\beta G^2m^5N(N-1)^2}{2}\right)^{\frac{3}{2}}\left[\frac{17}{3} - 3C - \ln\left(2\pi^2\beta N(N-1)^2G^2m^5\right)\right]. \quad (7)$$

This is the $N$ particle partition function for a highly interacting gravitational system from where we can calculate and analyze the thermodynamic equations of state. Before discussing the thermodynamic properties we see that the partition function becomes negative below a certain temperature what we call as the upper bound temperature $T_l$. It is given by

$$T = \frac{2\pi^2G^2}{k_B}N(N-1)^2m^5\exp\left(-\frac{17}{3}+3C\right). \quad (8)$$

The partition function is positive if the function $g(T)$ defined below is positive,

$$g(T) = 3\ln\left(2\pi^2\beta N(N-1)^2G^2m^5\right) - 17 + 9C. \quad (9)$$

In the plots of Figure 1 we can see behavior of $g(T)$ to obtain upper bound of temperature where it is positive. After analyzing the above part of the partition function, we found it is positive up to a temperature what we call upper bound and beyond that partition function becomes negative. It is obtained after using the natural approximation $G = k_B = m = 1$ and for the number of particles $N = 2$. We show that the upper bound of temperature is strongly depends on the number $N$.

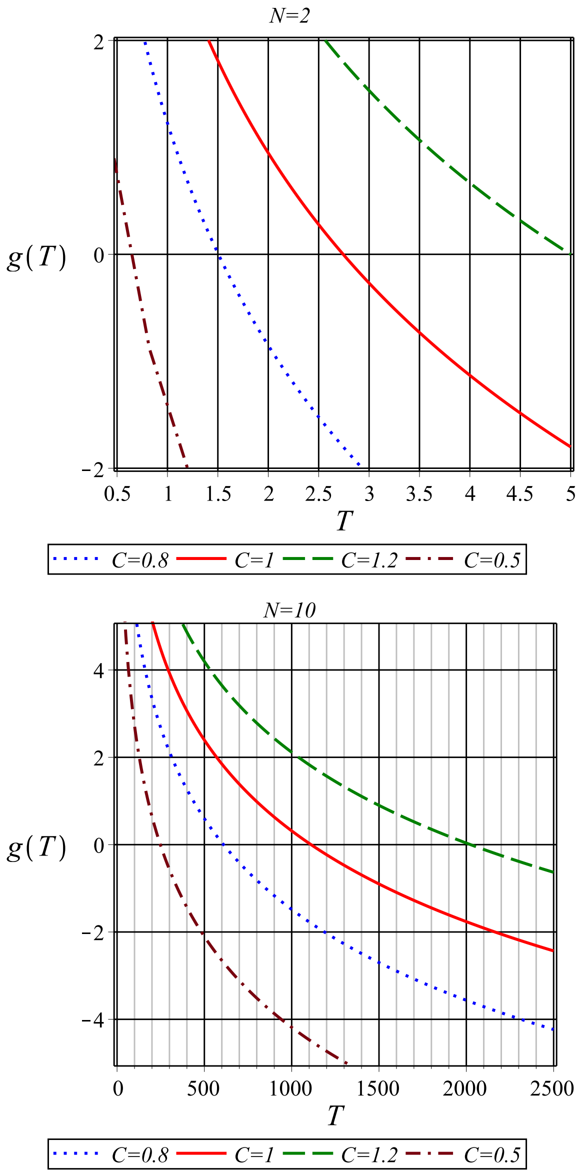


Figure 1: Upper bound of temperature for positive partition function for $G = m = 1$.

# 3 Internal energy and the interaction parameter

Internal energy $U$ is obtained using

$$U = -\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \beta}, \tag{10}$$

which is yield

$$U = \frac{3}{2\beta}\left[1 - \frac{2}{[17 - 9C - 3\ln\left(2\pi^2\beta N(N-1)^2G^2m^5\right)]}\right], \tag{11}$$

or we write it in the form

$$U = \frac{3}{2\beta}\left[1 - 2f_{in}\right], \tag{12}$$

where we define the interaction parameter $f_{in}$ as

$$f_{in} = \frac{1}{\left[17 - 9C - 3\ln\left(2\pi^2\beta N(N-1)^2G^2m^5\right)\right]}. \tag{13}$$

This takes care of the non ideal behavior of the system. For no interactions, it should be zero and for a complete virialized system $f_{in} = 1$, hence $0 \leq f_{in} \leq 1$. It helps us to obtain a lower bound of temperature as illustrated by Figure 2. The interaction parameter $f_{in}$ for strongly gravitating gas is plotted with respect to temperature to see the impact of temperature and we see the system becomes virialized that $f_{in}$ approaches 1 around a particular temperature. At very high temperatures the interactions become less dominant but do not vanish.

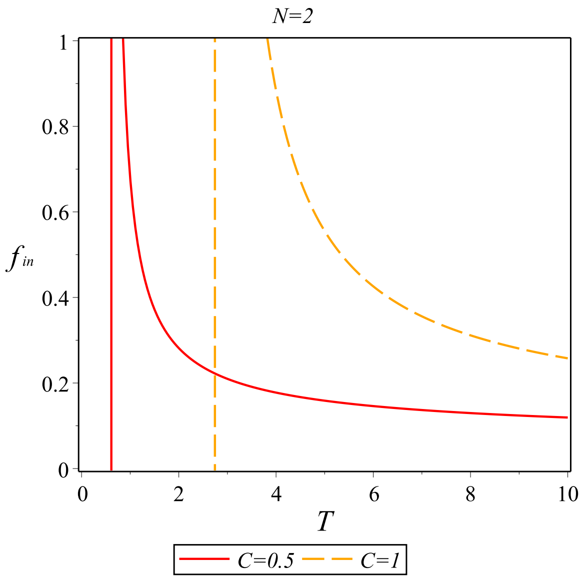


Figure 2: Interaction parameter in terms of temperature for $G = m = 1$.

From earlier theories for dilute systems clustering parameter $b$ has been defined as

$$b(N,V,T) = \frac{\frac{3G^2m^6}{2V}NT^{-3}}{1 + \frac{3G^2m^6}{2V}NT^{-3}}, \tag{14}$$

In Figure 3 we show behavior of the clustering parameter in terms of the temperature. Specific heat $C_v$ is obtained using

$$C_v = \left(\frac{\partial U}{\partial T}\right)_V. \tag{15}$$

Thus we have

$$C_v = \frac{3k_B}{2}\left(1 - 2f_{in} + 6f_{in}^2\right). \tag{16}$$

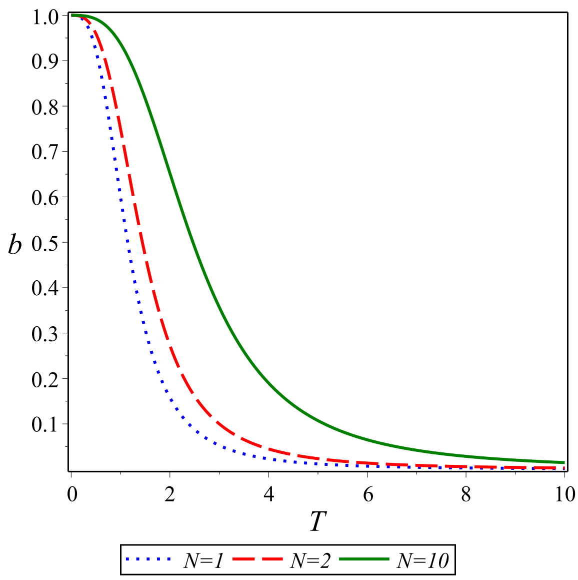


Figure 3: Clustering parameter in terms of temperature for $G = m = V = 1$.

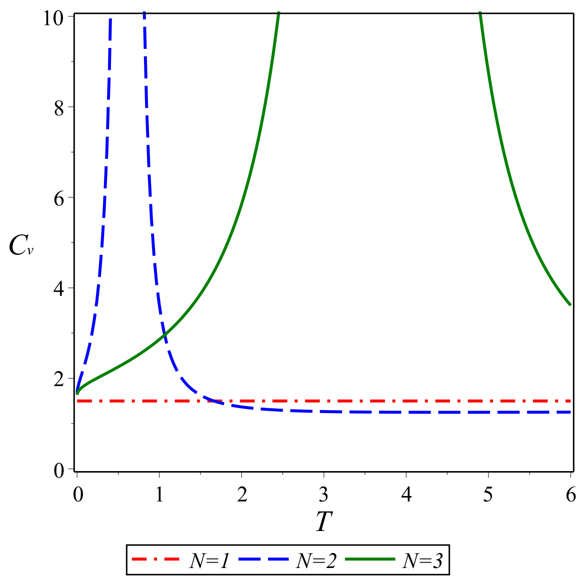


Figure 4: Specific heat (equation (16)) in terms of the temperature for $k_B = G = m = 1$.

In Figure 4 we can see typical behavior of the specific heat with temperature. In absence of interaction ($N = 1$) we can see that the specific heat is constant while effect of the interaction is a gap in a particular temperature.
The specific heat for dilute systems is obtained as

$$c_v = \frac{3k_b}{2}\left(1 + 4b - 6b^2\right). \tag{17}$$

In the plots of Figure 5 we compare specific heat given by the equation (16) with the equation (17) and find that they coincide at high temperatures. We can see that the specific heat for dilute system is negative at low temperatures, which yields a maximum and is reduced to a constant at high temperatures. The mentioned maximum of the specific heat is known as a Schottky anomaly.

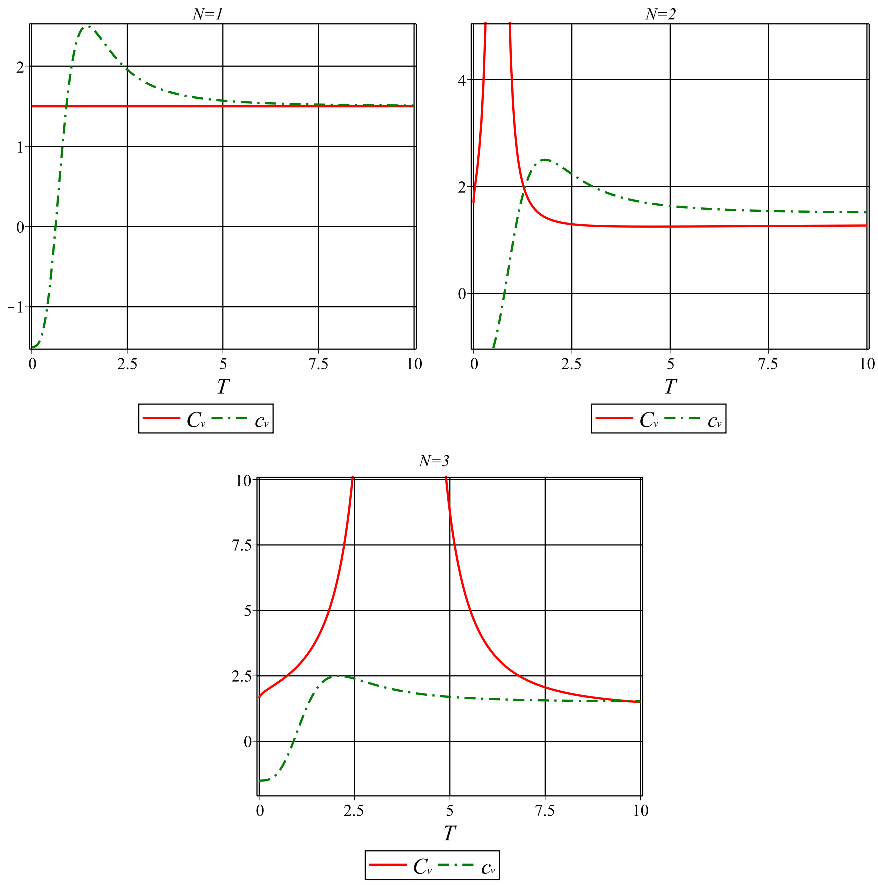


Figure 5: Temperature dependence of specific heat for a dilute system with $k_B = G = m = 1$.

## 4 Thermodynamic properties

Helmholtz free energy $F$ is obtained using

$$F = -\frac{1}{\beta} \ln \mathcal{Z}. \tag{18}$$

Thus, $F$ is obtained and written in terms of interaction parameter as follows,

$$F = -\frac{1}{\beta}\left[\frac{3}{2}\left(\frac{17}{3} - 3C - \frac{1}{3f_{in}}\right) - \ln(-f_{in}) - \ln(144\sqrt{2\pi}N!)\right]. \tag{19}$$

In Figure 6 we can see typical behavior of the Helmholtz free energy in terms of temperature.

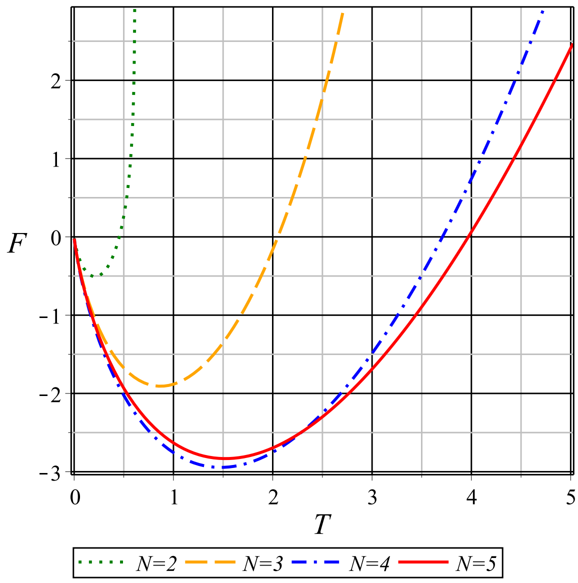


Figure 6: Helmholtz free energy in terms of the temperature for $k_B = G = m = 1$.

We can see a minimum which may be a sign of the model phase transition. It is indeed corresponding to phase transition point of the dilute systems which is illustrated by plots of Figure 5.
The entropy $S$ is obtained from $TS = U - F$, which yields,

$$S = \frac{3}{2\beta T}\left(1 - 2f_{in} + \frac{17}{3} - 3C - \frac{1}{3f_{in}} - \frac{2}{3}\ln(-f_{in}) - \frac{2}{3}\ln(144\sqrt{2\pi}N!)\right). \tag{20}$$

The chemical potential $\mu$ is obtained using

$$\mu = \left(\frac{\partial F}{\partial N}\right)_T, \tag{21}$$

which yields to the following relation,

$$\mu = \frac{1}{\beta}\left(\frac{\psi(N+1)}{N!} + \frac{3N^2 - 3N + 1}{N(N-1)^2}\left(f_{in} - \frac{3}{2}\right)\right), \tag{22}$$

where $\psi(z) = \frac{d\ln\Gamma(z)}{dz}$. Pressure $P$ is obtained using

$$PV = \frac{2N}{3}U, \tag{23}$$

which yields to the following corrected equation of state,

$$PV = \frac{N}{\beta}\left[1 - 2f_{in}\right]. \tag{24}$$

The grand canonical partition function $Z_G$ can be defined as

$$\ln Z_G = \beta PV, \tag{25}$$

which yields to the following relation,

$$Z_G = e^{N(1-2f_{in})}. \tag{26}$$

The fact that the interactions between different particles must cause some correlations in their positions. We study this by calculating the correlation function $\xi$. The integral of the correlation function over a certain volume in terms of the mean square fluctuation of the total number of particles in that volume is given by

$$\zeta \equiv \int \xi dV = \frac{<(\Delta N_i)^2>}{\bar{N}_i} - 1. \tag{27}$$

Expressing it in thermodynamic quantities as

$$\zeta \equiv \int \xi dV = -\frac{NT}{V}\left(\frac{\partial V}{\partial P}\right)_T - 1. \tag{28}$$

Calculating using the equation of pressure and using $k = 1$ we have

$$\zeta \equiv \int \xi dV = \frac{2f_{in}}{1-2f_{in}} \tag{29}$$

For a classical ideal gas $\int \xi dV = 0$ because there are no interactions. But we see the correlation function depends on interaction parameter and the number of particles. In Figure 7 we show typical behavior of the correlation function and see that it is a constant at high temperatures.

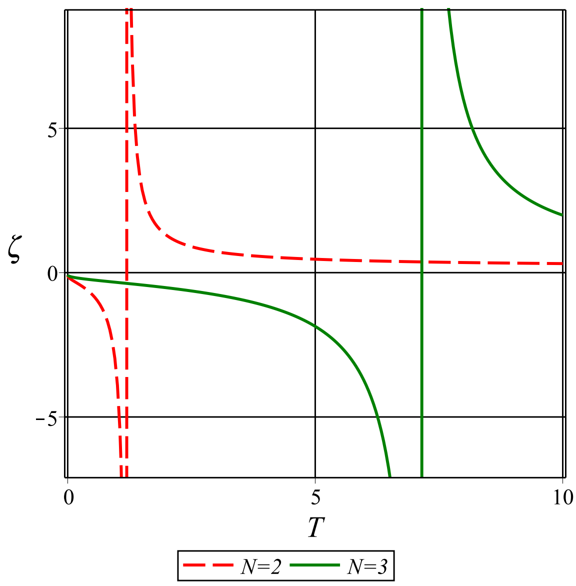


Figure 7: Correlation function in terms of the temperature for $k_B = G = m = 1$.

# 5 Moments generating function

We are going to calculate the moments $< r^k >$, where $k$ is a natural number. Its value in $\nu$ dimensions is given by the following formula

$$< r^k >_\nu = -\frac{1}{N!}\left[\frac{2\pi^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)}\right]^2 \int_0^\infty r^{\nu+k-1}dr \int_0^\infty p^{\nu-1}dp \exp\left(\beta\left(\frac{N(N-1)Gm^2}{2r}\right)\exp\left(-\frac{Np^2}{2m}\right)\right). \tag{30}$$

After evaluating the preceding integral, we obtain

$$< r^k >_\nu = -\frac{\sqrt{\pi}}{N!}2^{2-\frac{\nu}{2}}\pi^\nu \cos(\pi\nu)\left(\frac{m}{N\beta}\right)^{\frac{\nu}{2}}\left(\frac{N(N-1)^2\beta Gm^2}{2}\right)^{\nu+k} g(\Gamma), \tag{31}$$

where we defined

$$g(\Gamma) \equiv \frac{\Gamma(1+\nu)\Gamma(\nu)\Gamma(-\nu)}{\Gamma\left(\frac{\nu}{2}\right)^2\Gamma(1+k+\nu)\Gamma\left(\frac{\nu+1}{2}\right)}. \tag{32}$$

As we can see from formula (34), the value of $< r^k >$ is not defined in $\nu = 3$. To get its value at $\nu = 3$, we resorted again to Laurent's development. Then we have

$$< r^k >_3 = f(3)\left(-C + \frac{f'(3)}{f(3)}\right), \tag{33}$$

where

$$f(\nu) = -\frac{\sqrt{\pi}}{N!}2^{2-\frac{\nu}{2}}\pi^\nu \cos(\pi\nu)\left(\frac{m}{N\beta}\right)^{\frac{\nu}{2}}\left(\frac{N(N-1)^2\beta Gm^2}{2}\right)^{\nu+k} g'(\Gamma), \tag{34}$$

with

$$g'(\Gamma) \equiv \frac{\Gamma(1+\nu)\Gamma(\nu)}{\Gamma\left(\frac{\nu}{2}\right)^2\Gamma(1+k+\nu)\Gamma\left(\frac{\nu+1}{2}\right)}. \tag{35}$$

Hence, we can rewrite the equation (33) as following

$$< r^k >_3 = \frac{\sqrt{\pi}}{N!}\frac{2^{5/2}3\pi^2}{\Gamma(k+4)}\left(\frac{m}{N\beta}\right)^{\frac{3}{2}}\left(\frac{N(N-1)^2\beta Gm^2}{2}\right)^{3+k} X, \tag{36}$$

where

$$X \equiv -C + \ln\frac{\pi}{\sqrt{2}} + \ln\left(\frac{m}{N\beta}\right)^{\frac{1}{2}} + \ln\left(\frac{N(N-1)^2\beta Gm^2}{2}\right) - \frac{7}{6} - \frac{1}{3+k} + \Psi, \tag{37}$$

where

$$\Psi \equiv -\psi(3+k) + 2\psi(3) - 2\psi(\frac{3}{2}) - \psi(2). \tag{38}$$

The moments generating function is now

$$\mathcal{M}(t) = \sum_{k=0}^{\infty} < r^k > \frac{t^k}{k!}. \tag{39}$$

## 6 Distribution function

Distribution function is given by the following relation,

$$F(N) = \frac{z^N \mathcal{Z}}{Z_G}. \tag{40}$$

Making use of Stirlings approximation in chemical potential we find the fugacity $z$ as

$$z^N = e^{N\beta\mu}, \tag{41}$$

which yields to the following equation,

$$z^N = N^N e^{\left(\frac{3N^2-3N+1}{(N-1)^2}(f_{in}-3/2)\right)}. \tag{42}$$

Thus we get

$$F(N) = -\frac{N^N}{9N! f_{in}\sqrt{\pi}} e^{\left(\frac{3N^2-3N+1}{(N-1)^2}(f_{in}-3/2)-N(1-2f_{in})\right)} \left(\frac{\pi^2\beta G^2 m^5 N(N-1)^2}{2}\right)^{\frac{3}{2}} \tag{43}$$

In Figure 8 we draw distribution function for various temperatures.

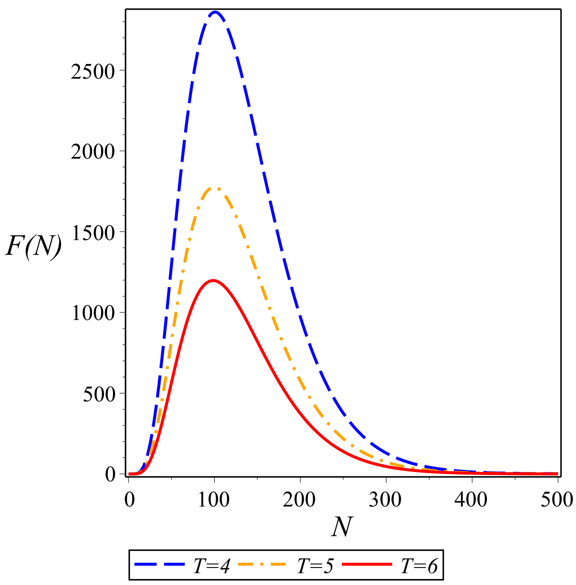


Figure 8: Distribution function in terms of the clustering number for $k_B = G = m = 1$.

## 7 Conclusions

In this paper, we have presented a theoretical investigation of the thermodynamic properties of strongly gravitating systems. By deriving the partition function and employing dimensional regularization techniques, we obtained expressions for various thermodynamic quantities, such as internal energy, specific heat, Helmholtz free energy, entropy, chemical potential, and pressure. The introduction of an interaction parameter allowed us to quantify the non-ideality of the system and study its effects on thermodynamic properties.

Our analysis revealed the existence of an upper bound temperature beyond which the partition function becomes negative, indicating a limitation of the model's validity. We also calculated the moments generating function, providing insights into the distribution of particle positions, and derived the distribution function for the system.

## Authors' Contributions

All authors have the same contribution.

## Data Availability

The manuscript has no associated data or the data will not be deposited.

## Conflicts of Interest

The authors declare that there is no conflict of interest.

## Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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