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Regular article

Thermodynamics of a Rotating Black Hole in Conformal Gravity

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Abstract. In this paper, we consider a metric of a rotating black hole in conformal gravity. We calculate the thermodynamical quantities for this rotating black hole including Hawking temperature and entropy in four dimensional space-time, as we obtain the effective value of Komar angular momentum. The result is valid on the event horizon of the black hole, and at any radial distance out of it. Also, we verify that the first law of thermodynamics will be held for this type of black hole.

Keywords: Black Hole; Thermodynamics; Conformal Gravity.

1 Introduction

During the last century, Einstein Gravity (EG) was one of the corner stones of theoretical physics. Despite of the success in the explanation of various gravitational phenomena in nature, there are some unsolved basic problems such as the singularity problem, black hole physics, and most importantly, quantum theory of gravity. There was an enormous effort in these lines to solve such problems but up to now, it has not been obtained a complete theory of gravity. One of such alternative theories of gravity is Conformal Gravity(CG) [1], the theoretical reason for which this extension is elegant in that, of all extensions that can be constructed by considering Lagrangians with more curvatures, the special case given by a conformally symmetric Lagrangian is unique, as discussed by Weyl; the phenomenological argument for which such an extension is important is that in this generalization the scale symmetry is related to the property of renormalizability [2]. Intuitively, besides local Lorentz symmetry, it also has an scaling symmetry in which the physics is invariant under the rescaling of the metric as $g_{\mu\nu} \longrightarrow e^{\Omega(x)} g_{\mu\nu}$. The observational fact for which this extension is interesting is that within this generalization, the projective structure gives rise to the possibility of describing in terms of background effects the rotation of galaxies, therefore reducing to geometry the problem of dark matter, as discussed by Mannheim and Kazanas in [3]. A detailed introduction on conformal gravity has been compiled in [4].

This paper is organized as follows: in section 2, we introduce the local solution of a neutral rotating black hole in pure conformal gravity. The main propose of this paper is calculating the thermodynamical quantities for this type of black holes. Black hole thermodynamics emerged from the classical general relativistic laws of black hole mechanics, summarized by Bardeen - Carter - Hawking, together with the physical insights by Bekenstein about black hole entropy [5] and the semiclassical derivation by Hawking of black hole evaporation. All the results obtained from 1963 to 1973 culminated in the famous four laws of black hole mechanics by Bardeen et al. [6]; therefor, in section 3, we obtain the thermodynamical quantities. We calculate some details in evaluating the temperature, entropy, and angular momentum, for the case of black hole considered in this paper. Hawking radiation results from the quantum effect of fields in a classical geometry with an event horizon. The flux of Hawking radiation can be also obtained through the scattering analysis and there have been the studies of the grey body factor for various black holes to calculate the Hawking temperature ($T = \frac{\kappa}{2\pi}$) [7]. To calculate the entropy of the black hole according to Bekenstine black hole entropy [5] there are several similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black hole area and entropy. In Wald formula for entropy had shown that this term is dependent on area of the black hole ($S = \frac{A}{4\pi G}$) [8]. In this paper, we show this fact with a different coefficient because of the lagrangian density that we use in Wald integral. At the end of this section, we calculate the effective value of Komar angular momentum by the spacelike killing vector of black hole and using the Hodge operators. In section 4, it is verified that the first law of thermodynamics will be held. The paper is concluded in section 5.

2 The metric

In this section, we briefly explain the metric solution of a black hole in conformal gravity [9]. Here, we start with the action

$$S_{CG} = -\alpha_g \int d^4x\sqrt{-g}C_{\mu\nu\lambda\theta}C^{\mu\nu\lambda\theta} + S_M = -2\alpha_g \int d^4x\sqrt{-g}[(R_{\mu\nu})^2 - \frac{1}{3}R^2] + S_M, \quad (1)$$

where

$$C_{\mu\nu\lambda\theta} = R_{\mu\nu\lambda\theta} + \frac{1}{6}R[g_{\mu\lambda}g_{\nu\theta} - g_{\mu\theta}g_{\lambda\nu}] - \frac{1}{2}[g_{\mu\lambda}R_{\nu\theta} - g_{\mu\theta}R_{\lambda\nu} - g_{\nu\lambda}R_{\mu\theta} + g_{\mu\theta}R_{\nu\lambda}],$$

is the Weyl conformal tensor. As a result the overall coupling of the theory $(-\alpha_g)$ is dimensionless which seems is a good news for UV finiteness of the theory [9]. After varying the action with respect to the metric, one obtains the equation of motion as

$$4\alpha_g W^{\mu\nu} = T_M^{\mu\nu}, \tag{2}$$

where $T_M^{\mu\nu}$ is Bach tensor and is defined as

$$W^{\mu\nu} = \frac{1}{3}\nabla_\mu\nabla_\nu R - \nabla_\lambda\nabla^\lambda R_{\mu\nu} + \frac{1}{6}(R^2 + \nabla_\lambda\nabla^\lambda R - 3(R_{\kappa\theta})^2)g_{\mu\nu} + 2R^{\kappa\theta}R_{\mu\kappa\nu\theta} - \frac{2}{3}RR_{\mu\nu}.$$

In addition, one finds that the matter part of the action should also respect to the scaling symmetry because the left hand side of the above equation is traceless so the matter part of the action should have a traceless energy-momentum tensor. Fortunately, by introducing a conformal coupling term for the scalar mass term the standard model Lagrangian is also conformable invariant [10]. In particular, it has obtained

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)^+(D^\mu\phi) - \frac{1}{12}R|\phi|^2 - \frac{\lambda}{4}|\phi|^4 - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}, \tag{3}$$

where

$$D_\mu = \nabla_\mu - ieA^a_\mu T_a,$$

and $F^a_{\mu\nu}$ is the Lie algebra valued field strength tensor of the gauge field. After solving the equation of motion for these fields, it has also obtained [9]

$$T_M^{\mu\nu} = \frac{1}{6}[g^{\mu\nu}\nabla^\lambda\nabla_\lambda|\phi|^2 - \nabla^\mu\nabla^\nu|\phi|^2 - G^{\mu\nu}|\phi|^2], \tag{4}$$

where $G^{\mu\nu}$ is Einstein tensor.

2.1 Rotating black hole

In this part, we use the slowly rotating solutions for pure conformal gravity, that obtained in [9]. Let us consider the following line element around a rotating black hole

$$ds^2 = \beta(r)dt^2 - \frac{dr^2}{\beta(r)} - r^2d\theta^2 - r^2\sin^2\theta(d\phi - \frac{N(r)}{r}dt)^2, \tag{5}$$

where

$$\beta(r) = C_1 + \frac{1}{3}\frac{C_1^2 - 1}{C_2 r} + C_2 r + C_3 r^2,$$

and

$$N(r) = C_4,$$

here, $C_1 = \sigma$ is considered as a constant of integration, $\frac{C_1^2-1}{C_2} = -m$, C_2 is the coefficient that appears in metric because of the CG solution, $C_3 = -\frac{\lambda}{3}$, that λ is the cosmological constant, and $N(r)$ is the constant value independent to r, $(N(r) = \omega)$; therefore, we can write the metric solution as a familiar form of

$$\beta(r) = \sigma - \frac{1}{3}\frac{m}{r} + Cr - \frac{\lambda}{3}r^2. \tag{6}$$

3 Thermodynamical quantities

In this section, we calculate the thermodynamical quantities of a rotating black hole with the metric in the previous section. We work in a system, that the value of $\hbar = G = C = 1$.

3.1 Singularity and area of the event horizon

The outer spherical boundary of the black hole, which is considered as its "surface", is called the event horizon. In this boundary, the velocity needed to escape exceeds the speed of light, which is the speed limit of the cosmos. Matter and radiation fall in, but they can't get out. Also, this radius depends on the mass of the black hole. In this part, first we obtain the black hole singularity by solving the equation $\beta(r) = 0$, so we can find the radius of black hole. This is a cubic equation that has three roots for r. Two of the roots are imaginary and for this reason they will be neglected. The other one is positive and the largest (r_+) and it gives the physical information that we want to obtain in this paper. After that, we obtain the area of the horizon for the black hole, which is of considerable importance because of the area theorem, which states that the horizon area of a classical black hole can never decrease in any physical process.

By setting $dr = dt = 0$ in the metric line elements, we can find line elements for the 2-dimensional horizon,

$$d\sigma^2 = -r_+^2 d\theta^2 - r_+^2 \sin^2\theta d\phi. \tag{7}$$

The area of the black hole horizon is then

$$A = \int_0^{2\pi} d\phi \int_0^{\pi} \sqrt{|\det\gamma|} d\theta, \tag{8}$$

where γ is the metric tensor for the black hole horizon.

3.2 Entropy

The entropy of black holes can be computed by the Wald formula [8]

$$S = -8\pi \int_{r=r_+} \sqrt{h} d^2x \epsilon_{ab}\epsilon_{cd} \frac{\partial \mathcal{L}}{\partial R_{abcd}}, \tag{9}$$

where h is the metric determinant on the surface.

In conformal gravity, we have [11]

$$\mathcal{L} = \frac{1}{2}\alpha C^2 = \frac{1}{2}\alpha(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 2R^{\mu\nu}R_{\mu\nu} + \frac{1}{3}R^2). \tag{10}$$

After some calculations, one can show that

$$\mathcal{L} = \frac{1}{2}\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2). \tag{11}$$

The indices a and b take the $(t,\, r)$ directions, thus we have

$$\epsilon_{ab}\epsilon_{cd}\frac{\partial \mathcal{L}}{\partial R_{abcd}} = \frac{1}{2}\alpha\left[-(g^{rr}R^{tt} + g^{tt}R^{rr}) + \frac{2}{3}g^{tt}g^{rr}R\right], \tag{12}$$

and

$$d^2x\sqrt{h}|_{r=r_+} = d\theta d\phi\sqrt{g_{\theta\theta}g_{\phi\phi}}|_{r=r_+}, \tag{13}$$

therefore, we find the entropy as follows

$$S = 4\pi\alpha\left[\frac{4m}{9r_+^3} - \frac{4}{3r_+^2}(1+\sigma) - \frac{2C}{r_+} + \frac{2}{9}\lambda(1+3r_+) + \frac{\omega^2}{3\pi r_+^2}\right]A, \tag{14}$$

where A is the area of the black hole.

3.3 Temperature

In this part, we attempt to obtain temperature of the aforementioned black hole. According to the Hawking radiation theorem, black hole temperature is dependent on surface gravity(κ), that it is equal to [12]

$$\kappa = \lim_{r\rightarrow r_+}\frac{\sqrt{a_\mu a^\mu}}{u^t}, \tag{15}$$

where

$$a^\mu = \Gamma^\mu_{\nu\lambda}u^\nu u^\lambda = (u^t)^2(\Gamma^\mu_{tt} + 2\Omega_H\Gamma^\mu_{t\varphi} + \Omega_H^2\Gamma^\mu_{\varphi\varphi}), \tag{16}$$

in which Ω_H, is the angular velocity of the black hole and equal to

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}}.$$

The normalization condition verifies that

$$1 = u^\mu u_\mu = (u^t)^2(g_{tt} + 2\Omega_H g_{t\varphi} + \Omega^2 g_{\varphi\varphi}). \tag{17}$$

So we obtain $a^\mu a_\mu$ as

$$a^2 = a^\mu a_\mu = |g^{rr}|(\partial_r \ln u^t)^2 + |g^{\theta\theta}|(\partial_\theta \ln u^t)^2, \tag{18}$$

where $u^t = \frac{1}{\beta(r)}$. Using the inverse metric coefficient

$$g^{rr} = -\beta(r), \tag{19}$$

$$g^{\theta\theta} = -\frac{1}{r^2}. \tag{20}$$

Since we are interested to obtain the surface gravity on the event horizon, so we only calculate the first term of Eq. (18), then as a result, the surface gravity on the event horizon is equal to

$$\kappa = \frac{1}{2}\beta'(r_+), \tag{21}$$

where

$$\beta'(r_+) = C_2 - \frac{1}{3}\frac{C_1^2 - 1}{C_2 r_+^2} + 2C_3 r_+. \tag{22}$$

By comparison Eqs. (21) and (22), we can calculate the surface gravity as

$$\kappa = \frac{1}{2}(\frac{1}{3}\frac{m}{r^2} - \frac{2}{3}\lambda r + C), \tag{23}$$

thus, the temperature is given by

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}(\frac{1}{3}\frac{m}{r^2} - \frac{2}{3}\lambda r + C). \tag{24}$$

3.4 Angular momentum

The Komar definition of the conserved quantity, corresponding to the spacelike Killing vector $\xi^{\mu}_{(\varphi)}$, in a coordinate free notation is given by [13]

$$K_{\eta} = \frac{1}{16\pi}\int *d\eta, \tag{25}$$

where

$$d\eta = \frac{\partial g_{03}}{\partial r} dr \wedge dt + \frac{\partial g_{03}}{\partial \theta} d\theta \wedge dt + \frac{\partial g_{33}}{\partial r} dr \wedge d\varphi + \frac{\partial g_{33}}{\partial \theta} d\theta \wedge d\varphi. \tag{26}$$

Instead of working with $dt, dr, d\theta, d\varphi$ we work with orthonormal one forms, so we write (26) as,

$$d\eta = \lambda_{10}\widehat{x_1} \wedge \widehat{x_0} + \lambda_{20}\widehat{x_2} \wedge \widehat{x_0} + \lambda_{13}\widehat{x_1} \wedge \widehat{x_3} + \lambda_{23}\widehat{x_2} \wedge \widehat{x_3}, \tag{27}$$

where

$$\begin{aligned}
\lambda_{10} &= -\frac{\partial g_{03}}{\partial r} - \frac{N(r)}{r}\frac{\partial g_{33}}{\partial r}, \\
\lambda_{20} &= -\frac{1}{r\sqrt{\beta(r)}}\frac{\partial g_{03}}{\partial \theta} - \frac{N(r)}{r}\frac{\partial g_{33}}{\partial \theta}, \\
\lambda_{13} &= \frac{\sqrt{\beta(r)}}{r\sin(\theta)}\frac{\partial g_{33}}{\partial r}, \\
\lambda_{23} &= \frac{1}{r^2\sin(\theta)}\frac{\partial g_{33}}{\partial \theta}.
\end{aligned} \tag{28}$$

The dual of (27) is [14]

$$*d\eta = \lambda_{10}\widehat{x_2} \wedge \widehat{x_3} + \lambda_{20}\widehat{x_1} \wedge \widehat{x_3} - \lambda_{13}\widehat{x_2} \wedge \widehat{x_0} - \lambda_{23}\widehat{x_1} \wedge \widehat{x_0}. \tag{29}$$

We can write (29) as

$$*d\eta = \delta_{rt} dr \wedge dt + \delta_{\theta t} d\theta \wedge dt + \delta_{r\varphi} dr \wedge d\varphi + \delta_{\theta\varphi} d\theta \wedge d\varphi, \tag{30}$$

where

$$\begin{aligned}
\delta_{\theta\varphi} &= \lambda_{10} r^2 \sin\theta, \\
\delta_{\theta t} &= -\lambda_{10} r N(r) \sin\theta, \\
\delta_{r\varphi} &= \lambda_{20}\frac{r\sin\theta}{\sqrt{\beta(r)}}, \\
\delta_{rt} &= -\lambda_{20}\frac{N(r)\sin\theta}{\sqrt{\beta(r)}} + \lambda_{23}, \\
\delta_{\theta t} &= \lambda_{13} r\sqrt{\beta(r)}.
\end{aligned} \tag{31}$$

To calculate Komar effective angular momentum, we need to define a boundary surface $(\partial\Sigma)$, that is characterized by a constant r and $dt = -\frac{g_{03}}{g_{00}} d\varphi$, so we have

$$*d\eta = -\frac{g_{03}}{g_{00}}\delta_{\theta t} d\theta \wedge dt + \delta_{\theta\varphi} d\theta \wedge d\varphi, \tag{32}$$

so we can write (25)as,

$$K_{\eta} = -\frac{1}{16\pi}\int \frac{g_{03}}{g_{00}}\delta_{\theta t} d\theta dt + \frac{1}{16\pi}\int \delta_{\theta\varphi} d\theta\ d\varphi, \tag{33}$$

Moving along a closed contour, the first term of the right hand side gives the shift of time between the initial and the final events. Since we are performing an integration over simultaneous events, this term must be subtracted from Eq. (33) ([15, 16]). So we write the Eq. (33) as follow

$$K_\eta = \frac{1}{16\pi}\int \lambda_{10} r^2 \sin\theta d\theta \; d\varphi. \tag{34}$$

By using Eq. (28)

$$K_\eta = \frac{1}{16\pi}\int (-\frac{\partial g_{03}}{\partial r} - \frac{N(r)}{r}\frac{\partial g_{33}}{\partial r}) r^2 \sin\theta d\theta d\varphi, \tag{35}$$

using the metric coefficient

$$g_{03} = rN(r)\sin^2\theta, \tag{36}$$

$$g_{33} = -r^2\sin^2\theta. \tag{37}$$

After calculating the integral (35) by using Eqs. (36), (37) we obtain the angular momentum as below

$$J = \frac{1}{6} r^2 \omega. \tag{38}$$

4 First law of thermodynamics

For perturbations of stationary black holes, the change of energy is related to the change of area, angular momentum, and electric charge according to the equations below

$$TdS = dE - dW, \tag{39}$$

where

$$dW = \Omega dJ + \Phi dQ. \tag{40}$$

Since entropy is dependent on the area of black hole, thus dS is proportional to dA :in addition, due to the energy of black hole is dependent on its mass, dE is proportional to dM; as a result, for (39) we have

$$dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ. \tag{41}$$

For this black hole, we have $\Phi = 0$ because it is neutral, and $\Omega = \frac{\omega}{r}$. Therefore, the first law of thermodynamics for this black hole is as follows

$$dM = \frac{\beta'(r_+)}{16\pi} dA + \frac{\omega}{r} dJ. \tag{42}$$

In conclusion we saw that the first law of black holes thermodynamics is held.

5 Conclusion

In this paper, we have used the metric of a rotating black hole, obtained in conformal gravity to calculate its thermodynamical quantities. We have calculated the Hawking temperature ($T = \frac{\kappa}{2\pi}$) by the formula ($\kappa = \lim_{r\to r_+} \frac{\sqrt{a_\mu a^\mu}}{u^t}$) and the entropy of a rotating black hole as the function of the area of the black hole by using the Lagrangin density for the metric in conformal gravity according to Wald formula and after that we have calculated the effective value of angular momentum with Komar expression for this black hole at any distance r, by choosing boundary of a finite spatial surface of radius r. This choice enabled us to evaluate the Komar integrals without any asymptotic approximation. At the end we have shown that the first law of the thermodynamics is held for this black hole.

Authors' contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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