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Regular article

𝔮-discrete Painlevé VI equations from M2-branes

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Abstract. In this paper, we review the novel connection between a theory of N M2-branes on $(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}/\mathbb{Z}_2)/\mathbb{Z}_k$ and a discrete integrable system. Besides the IR duality induced by the Hanany-Witten transitions in the type IIB brane construction, the Fermi gas formalism tells us that the partition function of this theory enjoys a larger discrete symmetry which is the Weyl group $W(D_5)$ of $D_5 = \mathrm{SO}(10)$. The Fermi gas formalism, together with the topological string/spectral theory correspondence and the connection between the integrable systems and the Nekrasov partition functions recently found, further suggests that the grand partition function of this M2-brane partition function satisfies a bilinear difference equation associated with $W(D_5)$, called 𝔮-deformed Painlevé VI. By using the exact values of the partition functions we identify the explicit expression of the bilinear equations and confirm that these equations are indeed satisfied for higher order in the chemical potential dual to the rank N. This article is based on [1] and [2].

Keywords: M2-branes, M-theory, Dualities, Matrix model, Painlevé equation.

1 Introduction

One of the motivations to study supersymmetric gauge theories is to find new aspects of quantum field theories with the help of the solvability of the models. To pursue this direction it is important to choose a model which is sufficiently complex so that it would enjoy some non-trivial mathematical structures to be revealed, and highly tractable at the same time.

Among other examples, a class of the $\mathcal{N} \geq 4$ three-dimensional quiver Chern-Simons matter theories [3–8] fits well with these requirements. Each of these theories describes a stack of M2-branes in M-theory on a certain background geometry, hence we may expect that these theories have rich mathematical structure related to M-theory. On the other hand, the supersymmetry allows us to reduce the path integral for the partition function $Z(N)$ to an $\mathcal{O}(N)$-dimensional ordinary integration by the supersymmetry localization technique [9, 10]. This $\mathcal{O}(N)$ dimensional integration can be evaluated by the standard techniques in the analysis of the matrix models such as the large N saddle point approximation and the 't Hooft expansion. Indeed by the large N saddle point approximation the large N leading behavior of the free energy $-\log Z(N)$ was obtained [11], which reproduces the $N^{3/2}$ scaling behavior argued in the gravity side [12]. Moreover, for these theories there is another powerful way to evaluate the partition function called Fermi gas formalism [13], which is not available for other general supersymmetric theories.[1] Indeed, various non-trivial structures in the large N expansion of the partition function were revealed by using the Fermi gas formalism. For example, all order $1/N$ perturbative corrections to the free energy were found to add up to the Airy function characterized only by three parameters.[2] Furthermore, the infinite series of the $1/N$ non-perturbative corrections are also given by the large radius expansion of the free energy of refined topological string on a certain Calabi-Yau threefold associated with the background orbifold of the M2-branes [17, 18].

The Fermi gas formalism is not only a powerful computational tool, but also provides a new connection between the theories of M2-branes and quantum algebraic curves. The fact that all order $1/N$ perturbative corrections add up to the Airy function follows immediately from the correspondence with a curve. The curve is also the mirror curve of the target Calabi-Yau threefold of the topological string which precisely encodes the Calabi-Yau threefold including its moduli parameters.

From the point of view of the quantum curve, the IR duality of the original theory of M2-branes is realized as the discrete symmetry of the moduli space of the curve induced by the coordinate transformations. In general, the symmetry of the curve is larger than the known IR duality (see e.g. [19]) and hence the picture of the quantum curve tells us the new symmetry of the theory. Furthermore, recently it was found that the partition function of the ABJ theory, the $\mathrm{U}(N_1)_k \times \mathrm{U}(N_2)_{-k}$ circular quiver superconformal Chern-Simons theory describing $\min(N_1, N_2)$ M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ [5–8], satisfies an infinitely many bilinear relations among different values of N_1, N_2 [20]. In terms of the grand canonical partition function, these relations are equivalent to one of the $\mathfrak{q}$-deformed Painlevé equations called $\mathfrak{q}$-Painlevé III$_3$ ($\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$). The $\mathfrak{q}$-deformed Painlevé equations are classified by the discrete symmetries of genus-one curves (or the curves themselves) [21], and $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$ coincides with the one associated with the symmetry of the curve (or the curve itself) of the Fermi gas formalism of the ABJ theory [22]. In [1, 2], we found another more non-trivial example

[1]Note that the existence of the Fermi gas formalism and the subsequent properties of the partition functions were also confirmed for a slightly larger class of the Chern-Simons matter theories such as the circular quiver theories with $\mathcal{N} = 3$ and the theories with affine $\widehat{D}$-type quiver diagram [14, 15]. However, for simplicity in this article, we focus on the $\mathcal{N} = 4$ theories.

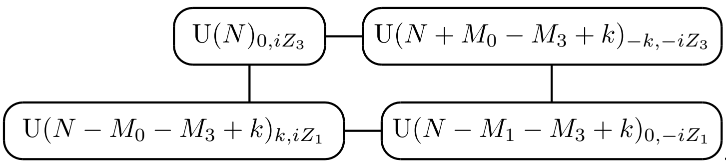
[2]Historically the large N leading behavior and the resummation of the all order $1/N$ perturbative corrections were first obtained through the 't Hooft expansion [16].

for this connection. Namely, we consider the the $\mathrm{U}(N_1)_k \times \mathrm{U}(N_2)_0 \times \mathrm{U}(N_3)_{-k} \times \mathrm{U}(N_4)_0$ circular quiver superconformal Chern-Simons theory which describes $\min(N_1, N_2, N_3, N_4)$ M2-branes on $(\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_k$. The curve associated with the Fermi gas formalism of this theory is the genus-one curve with $W(D_5)$ symmetry acting on the five-dimensional moduli space. The moduli parameters are identified with the three relative ranks of the four-node quiver together with the two degrees of freedom of the Fayet-Illiopoulos parameters. Hinted with the dictionary between the moduli parameters of the curves and the three-dimensional parameters as well as the exact expressions for the partition functions at the lowest ranks, in [1, 2] we have identified the explicit expressions for the discrete bilinear relations for the grand partition function of this four-node super Chern-Simons theory and also provides non-trivial evidences that the relations are indeed satisfied also for higher order in the fugacity dual to the lowest rank. These bilinear relations are the same as the Hirota bilinear form of the $\mathfrak{q}$-deformed Painlevé VI equation ($\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$), which is associated with the $W(D_5)$ symmetry under the Sakai classification [21, 23].

This review is organized as follows. In the next section we introduce the model of our concern, the four-node quiver superconformal Chern-Simons theory, and fix the notation for the parameters and the overall normalization of the partition function. In section 3 we explain the Fermi gas formalism for the partition function and identify the quantum curve associated with this theory. Although it is difficult to construct the Fermi gas formalism and invert the density operator for general values of the relative ranks and the FI parameters, under a few assumptions we can skip such steps and obtain the quantum curve directly by the interpolations from special points, as we review in section 6. In section 4 we explain the relation between the grand partition function and $\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$. In section 5 we summarize and discuss possible future directions.

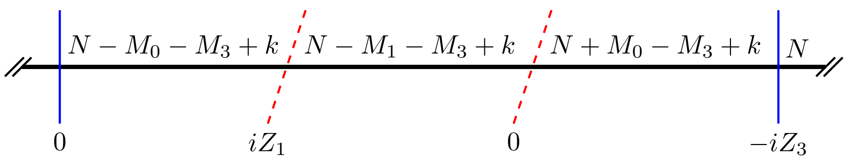
2 The model

The model we consider in this article is the three-dimensional $\mathcal{N} = 4$ circular quiver supersymmetric Chern-Simons matter theory given by the following quiver diagram



. (1)

The theory consists of the $\mathrm{U}(n_i)$ vector multiples with Chern-Simons action with level k_i and the Fayet-Illiopoulos parameter ζ_i associated with each node labeled by $\mathrm{U}(n_i)_{k_i,\zeta_i}$, and pairs of $\mathrm{U}(n_i) \times \mathrm{U}(n_j)$-bifundamental chiral multiplet associated with each bond connecting two nodes labeled by $\mathrm{U}(n_i)_{k_i,\zeta_i}$ and $\mathrm{U}(n_j)_{k_j,\zeta_j}$. In the following, we use the abbreviation $\boldsymbol{M} = (M_0, M_1, M_3, Z_1, Z_3)$ for the three relative ranks and two Fayet-Illiopoulos parameters. This theory can be realized by the following brane setup in type IIB superstring theory [3, 4, 24, 25]



, (2)

where the horizontal lines are stacks of D3-branes, the vertical blue lines are NS5-branes and the dashed red lines are $(1,k)$5-branes stretched in the following directions:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	–	–	–	–						
NS5	–	–	–		–		–		–	
$(1,k)5$	–	–	–		θ_{45}		θ_{67}		θ_{89}	

, (3)

where θ_{ij} stands for the direction in ij-plane tilted from the i-axis by an angle $\arctan k$. Each stack of D3-branes stretched between a pair of 5-branes corresponds to a gauge node, where the Chern-Simons level is given by the difference of the two D5-charges of the 5-branes [24]. Each 5-brane is also assigned with the Fayet-Illiopoulos parameter which are written below the 5-branes in the figure, with which the Fayet-Illiopoulos parameter on the gauge node is given by the difference of the two Fayet-Illiopoulos parameters.

The partition function of the theory (1) can be calculated by the supersymmetry localization formula as [10]

$$\begin{aligned}Z_{k,\boldsymbol{M}}(N) &= \frac{e^{iP_{k,\boldsymbol{M}}(N)}}{N_1!N_2!N_3!N_4!}\int\prod_{i=1}^{N_1}\frac{d\lambda_i^{(1)}}{2\pi}\prod_{i=1}^{N_2}\frac{d\lambda_i^{(2)}}{2\pi}\prod_{i=1}^{N_3}\frac{d\lambda_i^{(3)}}{2\pi}\prod_{i=1}^{N_4}\frac{d\lambda_i^{(4)}}{2\pi}\\ &\times e^{\frac{ik}{4\pi}\sum_{i=1}^{N_1}(\lambda_i^{(1)})^2-\frac{ik}{4\pi}\sum_{i=1}^{N_2}(\lambda_i^{(2)})^2}e^{Z_1(\sum_{i=1}^{N_1}\lambda_i^{(1)}-\sum_{i=1}^{N_2}\lambda_i^{(2)})-Z_3(\sum_{i=1}^{N_3}\lambda_i^{(3)}-\sum_{i=1}^{N_4}\lambda_i^{(4)})}\\ &\times\prod_{a=1}^{4}\frac{\prod_{i<j}^{N_a}(2\sinh\frac{\lambda_i^{(a)}-\lambda_j^{(a)}}{2})^2}{\prod_{i=1}^{N_a}\prod_{j=1}^{N_{a+1}}2\cosh\frac{\lambda_i^{(a)}-\lambda_j^{(a+1)}}{2}},\end{aligned} \tag{4}$$

where $N_1 = N - M_0 - M_3 + k$, $N_2 = N - M_1 - M_3 + k$, $N_3 = N + M_0 - M_3 + k$ and $N_4 = N$. Note that N_5 and $\lambda_i^{(5)}$ are identified with N_1 and $\lambda_i^{(1)}$. We have also introduced an overall phase $e^{iP_{k,\boldsymbol{M}}(N)}$

$$\begin{aligned}e^{iP_{k,\boldsymbol{M}}(N)} &= \exp\Biggl[i\pi\Bigl(M_0N+\frac{M_0^3-M_0}{3k}+\frac{M_0(M_1^2+M_3^2)}{2k}-2M_0M_3+\frac{3kM_0}{2}-2M_1(Z_1+Z_3)\\ &+\frac{(M_0-M_1)Z_1^2}{k}+\frac{(M_0-M_3+k)Z_3^2}{k}+\frac{(M_0-M_1-M_3-k)Z_1Z_3}{k}\Bigr)\Biggr]\end{aligned} \tag{5}$$

by hand so that the coefficients of the $\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$ bilinear relations simplify as (75) below.

3 Fermi gas formalism and quantum curve

The partition function (4) is conjectured to be rewritten into the following form [1, 26]

$$Z_{k,\boldsymbol{M}}(N) = \frac{Z_{k,\boldsymbol{M}}(0)}{N!}\int\frac{d^Nx}{(2\pi)^N}\det_{i,j}[\langle x_i|\hat{\rho}_{\boldsymbol{M}}|x_j\rangle], \tag{6}$$

with some one-dimensional quantum mechanical operator $\hat{\rho}_{\boldsymbol{M}}$ which depends on $\boldsymbol{M} = (M_0, M_1, M_3, Z_1, Z_3)$ as well as k. Here we have introduced the following notation for one-dimensional quantum mechanics:

$$
\begin{aligned}
&|x\rangle\text{: eigenstate of position operator } \hat{x},\\
&|p\rangle\!\rangle\text{: eigenstate of momentum operator } \hat{p},\\
&[\hat{x},\hat{p}]=2\pi i k,\\
&\langle x|y\rangle=2\pi\delta(x-y),\quad \langle\!\langle p|p'\rangle\!\rangle=2\pi\delta(p-p'),\\
&\langle x|p\rangle\!\rangle=\frac{1}{\sqrt{k}}e^{\frac{ixp}{2\pi k}},\quad \langle\!\langle p|x\rangle=\frac{1}{\sqrt{k}}e^{-\frac{ixp}{2\pi k}}.
\end{aligned}
\tag{7}
$$

Note that the right-hand side of (6), up to the overall factor $Z_{k,\boldsymbol{M}}(N=0)$, is of the same form as the partition function of N-particle ideal Fermi gas with single-particle Hamiltonian given by $-\log\hat{\rho}_{\boldsymbol{M}}$, hence this rewriting is called Fermi gas formalism [13].

When $(M_0,M_1,M_3)=(0,0,k)$, namely, $N_1=N_2=N_3=N_4=N$, the fermi gas formalism (6) can be derived relatively easily by using the Cauchy determinant formula

$$
\frac{\prod_{i<j}^{N}2\sinh\frac{x_i-x_j}{2k}\prod_{i<j}^{N}2\sinh\frac{y_i-y_j}{2k}}{\prod_{i,j}^{N}2\cosh\frac{x_i-y_j}{2k}}=k^N\det_{i,j}\Bigl[\frac{1}{2k\cosh\frac{x_i-y_j}{2k}}\Bigr]=k^N\det_{i,j}\Bigl[\langle x_i|\frac{1}{2\cosh\frac{\hat{p}}{2}}|x_j\rangle\Bigr],
\tag{8}
$$

and the Cauchy-Binet formula

$$
\frac{1}{N!}\int d^Nz\det_{i,j}[f_i(z_j)\det_{i,j}[g_i(z_j)]=\det_{i,j}\Bigl[\int dz f_i(z)g_j(z)\Bigr].
\tag{9}
$$

Here in the rightmost side of (8) we have used the notations for one-dimensional quantum mechanics (7). The derivation goes as follows. First applying the Cauchy determinant formula (8) to the one-loop determinant factors in (4) we obtain

$$
\begin{aligned}
&Z_{k,(0,0,k,Z_1,Z_3)}(N)\\
&=\frac{e^{-2\pi i Z_1Z_3}}{(N!)^4}\int\frac{d^N\lambda_i^{(1)}}{(2\pi)^N}\frac{d^N\lambda_i^{(2)}}{(2\pi)^N}\frac{d^N\lambda_i^{(3)}}{(2\pi)^N}\frac{d^N\lambda_i^{(4)}}{(2\pi)^N}e^{\frac{i}{4\pi k}\sum_{i=1}^N(\lambda_i^{(1)})^2}e^{\frac{Z_1}{k}\sum_{i=1}^N\lambda_i^{(1)}}\\
&\times\det_{i,j}\Bigl[\langle\lambda_i^{(1)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}|\lambda_i^{(2)}\rangle\Bigr]e^{-\frac{Z_1}{k}\sum_{i=1}^N\lambda_i^{(2)}}\\
&\times\det_{i,j}\Bigl[\langle\lambda_i^{(2)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}|\lambda_i^{(3)}\rangle\Bigr]e^{-\frac{i}{4\pi k}\sum_{i=1}^N(\lambda_i^{(3)})^2}e^{-\frac{Z_3}{k}\sum_{i=1}^N\lambda_i^{(3)}}\\
&\times\det_{i,j}\Bigl[\langle\lambda_i^{(3)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}|\lambda_i^{(4)}\rangle\Bigr]e^{\frac{Z_3}{k}\sum_{i=1}^N\lambda_i^{(4)}}\det_{i,j}\Bigl[\langle\lambda_i^{(4)}|\frac{1}{2\cosh\frac{\hat{p}}{2}}|\lambda_i^{(1)}\rangle\Bigr].
\end{aligned}
\tag{10}
$$

Now we can convolute the determinants by applying the Cauchy-Binet formula (9) to obtain

$$
\begin{aligned}
&Z_{k,(0,0,k,Z_1,Z_3)}(N)\\
&=\frac{e^{-2\pi i Z_1Z_3}}{N!}\int\frac{d^Nx}{(2\pi)^N}\\
&\det_{i,j}\Bigl[\langle x_i|e^{\frac{i}{4\pi k}\hat{x}^2}e^{\frac{Z_1}{k}\hat{x}}\frac{1}{2\cosh\frac{\hat{p}}{2}}e^{-\frac{Z_1}{k}\hat{x}}\frac{1}{2\cosh\frac{\hat{p}}{2}}e^{-\frac{i}{4\pi k}\hat{x}^2}e^{-\frac{Z_3}{k}\hat{x}}\frac{1}{2\cosh\frac{\hat{p}}{2}}e^{\frac{Z_3}{k}\hat{x}}\frac{1}{2\cosh\frac{\hat{p}}{2}}|x_j\rangle\Bigr].
\end{aligned}
\tag{11}
$$

This is precisely of the form (6). After a similarity transformation, $\hat{\rho}_{(0,0,k,Z_1,Z_3)}$ can be written as

$$
\hat{\rho}_{(0,0,k,Z_1,Z_3)}=\frac{1}{2\cosh\frac{\hat{x}}{2}}\frac{1}{2\cosh\frac{\hat{p}+2\pi i Z_1}{2}}\frac{1}{2\cosh\frac{\hat{p}}{2}}\frac{1}{2\cosh\frac{\hat{x}-2\pi i Z_3}{2}}.
\tag{12}
$$

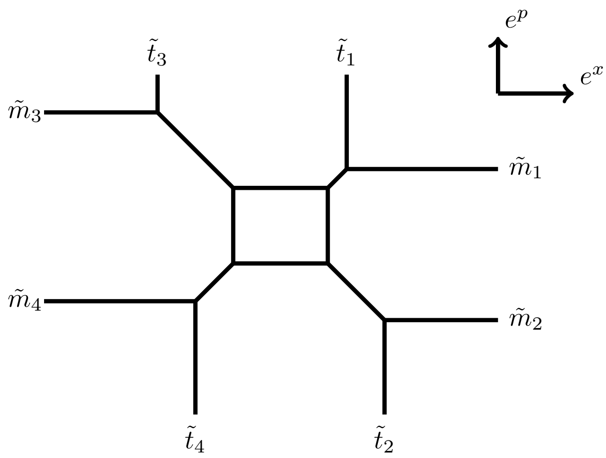
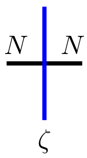
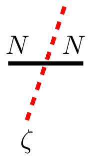


Figure 1: The curve $\rho_{\boldsymbol{M}}^{-1}(x,p) = const.$ of the four-node quiver Chern-Simons theory (1).

Notice that this density matrix can be obtained directly from the type IIB brane setup (2) through the following assignments of operators to each 5-branes



$$\rightarrow \frac{1}{2\cosh\frac{\hat{x}+2\pi\zeta}{2}},$$



$$\rightarrow \frac{1}{2\cosh\frac{\hat{p}+2\pi\zeta}{2}}. \tag{13}$$

This formula holds also for the setups containing any number of NS5-branes and $(1,k)$5-branes in any order, as long as the number of D3-branes is the same for all segments.

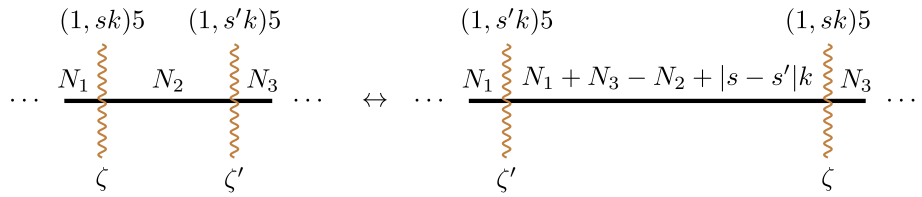
The inverse of $\hat{\rho}_{(0,0,k,Z_1,Z_3)}$, which we shall call "quantum curve", takes the following form

$$\hat{\rho}_{(0,0,k,Z_1,Z_3)}^{-1} = \sum_{m=-1,0,1}\sum_{n=-1,0,1} c_{mn} e^{m\hat{x}+n\hat{p}}. \tag{14}$$

Indeed, the classical equation $\rho_{(0,0,k,Z_1,Z_3)}(x,p)^{-1} = \sum_{m,n} c_{mn} e^{mx+np} = const.$ defines a genus one curve with eight asymptotic points, as depicted in figure 1. Here we have defined the classical curve by the Weyl ordering according to the prescription in [18]. The asymptotic loci are obtained from c_{mn}'s by solving the quadratic equations in e^p obtained by sending $x \to \pm\infty$ or the equations obtained by sending $p \to \pm\infty$. For example, the two asymptotic loci $e^p = \tilde{m}_1, \tilde{m}_2$ at $x = \infty$ are the two solutions of $c_{1,1}e^p + c_{1,0} + c_{1,-1}e^{-p} = 0$.

Besides the overall rescaling and the constant $c_{0,0}$, the curve has five moduli, which correspond to the eight asymptotic loci up to the trivial translations $x \to x + const.$, $p \to p + const.$ and an additional constraint due to Vieta's formula, $8 - 2 - 1 = 5$. There are several pieces of evidence that the five-dimensional moduli space is spanned by turning on the three rank deformations (M_0, M_1, M_3) and the two FI parameters (Z_1, Z_2) [1, 26–31]. Although it is difficult to construct the Fermi gas formalism (and then calculate the inverse of $\hat{\rho}_{\boldsymbol{M}}$) for general rank deformations, we can identify the relation between the rank/FI parameters and the moduli of the curve by interpolation from special points. Namely, a type IIB brane configuration can be transformed to another brane configuration by the

Hanany-Witten transition [32, 33]



. (15)

When the rank differences are some special values proportional to k, the resulting brane configuration has a uniform rank, where $\hat{\rho}_{\boldsymbol{M}}$ and its inverse can be obtained straightforwardly by applying the rule (13). Since the Hanany-Witten transition induces an IR duality, we conclude that the quantum curve thus obtained also gives the quantum curve for the configuration before the Hanany-Witten transition.[3]

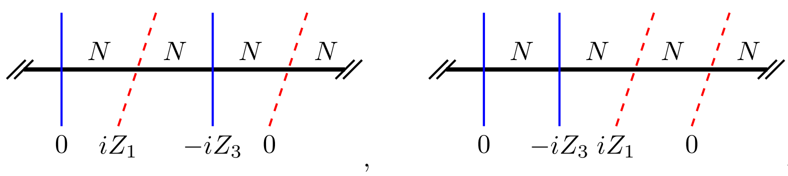
Let us first figure out the values of $\tilde{m}_i, \tilde{t}_i$ for $(M_0, M_1, M_3) = (0, 0, k)$, where $\hat{\rho}_{(0,0,k,Z_1,Z_3)}$ is given by (12). It is straightforward to calculate the inverse of $\hat{\rho}_{(0,0,k,Z_1,Z_3)}$ and organize it in the form (14), as

$$\begin{aligned}\hat{\rho}^{-1}_{(0,0,k,Z_1,Z_3)} &= \Big(2\cosh\frac{\hat{x}-2\pi i Z_3}{2}\Big)\Big(2\cosh\frac{\hat{p}}{2}\Big)\Big(2\cosh\frac{\hat{P}+2\pi i Z_1}{2}\Big)\Big(2\cosh\frac{\hat{x}}{2}\Big)\\ &= e^{\pi i(Z_1-Z_3)}e^{\hat{x}+\hat{p}} + e^{\pi i Z_1}(2\cos\pi(Z_3-k))e^{\hat{p}} + e^{\pi i(Z_1+Z_3)}e^{-\hat{x}+\hat{p}}\\ &\quad + e^{-\pi i Z_3}(2\cos\pi Z_1)e^{\hat{x}} + 4\cos\pi Z_1\cos\pi Z_3 + e^{\pi i Z_3}(2\cos\pi Z_1)e^{-\hat{x}}\\ &\quad + e^{\pi i(-Z_1-Z_3)}e^{\hat{x}-\hat{p}} + e^{-\pi i Z_1}(2\cos\pi(Z_3+k))e^{-\hat{p}} + e^{\pi i(-Z_1+Z_3)}e^{-\hat{x}-\hat{p}}. \end{aligned} \quad (16)$$

From this we obtain

$$\begin{aligned}\rho_{(0,0,k,Z_1,Z_3)}(x,p)^{-1}|_{x\to\infty} = 0 \to e^p = \tilde{m}'_1, \tilde{m}'_2 = -e^{-2\pi i Z_1}, -1,\\ \rho_{(0,0,k,Z_1,Z_3)}(x,p)^{-1}|_{x\to-\infty} = 0 \to e^p = \tilde{m}'_3, \tilde{m}'_4 = -1, -e^{-2\pi i Z_1},\\ \rho_{(0,0,k,Z_1,Z_3)}(x,p)^{-1}|_{p\to\infty} = 0 \to e^x = \tilde{t}'_1, \tilde{t}'_3 = -e^{\pi i(2Z_3-k)}, -e^{\pi i k},\\ \rho_{(0,0,k,Z_1,Z_3)}(x,p)^{-1}|_{p\to-\infty} = 0 \to e^x = \tilde{t}'_2, \tilde{t}'_4 = -e^{-\pi i k}, -e^{\pi i(2Z_3+k)}, \end{aligned} \quad (17)$$

where we have denoted the asymptotic loci with prime, $\tilde{m}'_i, \tilde{t}'_i$, taking into account the gauge degrees of freedom of the overall rescalings corresponding to $x \to x + cosnt.$ and $p \to p + const.$, which we shall fix below (21). Next, let us consider the following two brane configurations



. (18)

Since these configurations are of uniform ranks, we can apply the formulas (13) to write down $\hat{\rho}$ and read off $\tilde{m}'_i, \tilde{t}'_i$ from the coefficients in $\hat{\rho}^{-1}$. On the other hand, these configurations

[3] Precisely speaking, we can prove when the 5-branes are of a different kind [34], and also argue when the 5-branes are of the same kind [35], that the partition functions before and after the brane exchange are different only by an overall factor independent of the overall rank N. Hence we assume that the density matrix $\hat{\rho}_{\boldsymbol{M}}$ of the Fermi gas formalism, which is defined after normalizing the partition function by the N-independent part $Z_{k,\boldsymbol{M}}(0)$, is the same for the two brane configurations (15).

can be transformed, by moving the NS5-brane with FI parameter $-iZ_3$ to right under the Hanany-Witten rule (15), into the configuration with the original ordering of 5-branes (2) with $(M_0, M_1, M_3) = (\frac{k}{2}, \frac{k}{2}, \frac{k}{2})$ and $(M_0, M_1, M_3) = (k, 0, 0)$ respectively. In summary, now we have obtained the values of the asymptotic loci at three special points:

(M_0, M_1, M_3)	$\tilde{m}'_1$	$\tilde{m}'_2$	$\tilde{m}'_3$	$\tilde{m}'_4$	$\tilde{t}'_1$	$\tilde{t}'_2$	$\tilde{t}'_3$	$\tilde{t}'_4$
$(0, 0, k)$	$-e^{-2\pi iZ_1}$	-1	-1	$-e^{-2\pi iZ_1}$	$-e^{\pi i(2Z_3-k)}$	$-e^{-\pi ik}$	$-e^{\pi ik}$	$-e^{\pi i(2Z_3+k)}$
$(\frac{k}{2}, \frac{k}{2}, \frac{k}{2})$	$-e^{-2\pi iZ_1}$	$-e^{\pi ik}$	$-e^{-\pi ik}$	$-e^{-2\pi iZ_1}$	$-e^{2\pi iZ_3}$	$-e^{-\pi ik}$	$-e^{\pi ik}$	$-e^{2\pi iZ_3}$
$(k, 0, 0)$	$-e^{\pi i(-2Z_1+k)}$	$-e^{\pi ik}$	$-e^{-\pi ik}$	$-e^{\pi i(-2Z_1-k)}$	$-e^{\pi i(2Z_3+k)}$	$-e^{-\pi ik}$	$-e^{\pi ik}$	$-e^{\pi i(2Z_3-k)}$

(19)

where we have chosen the ordering of $\tilde{m}'_{1,2}$, $\tilde{m}'_{3,4}$, $\tilde{t}'_{1,3}$ and $\tilde{t}'_{2,4}$ for $(M_0, M_1, M_3) = (\frac{k}{2}, \frac{k}{2}, \frac{k}{2})$ and $(M_0, M_1, M_3) = (k, 0, 0)$ so that the Z_1, Z_3-dependences of the asymptotic loci are the same as those for $(M_0, M_1, M_3) = (0, 0, k)$. By assuming that the M_0, M_1, M_3-dependences of the loci are simple exponential functions $e^{\pi i(aM_0+bM_1+cM_3)}$, we can interpolate (19) to general M_0, M_1, M_3 by determining a, b, c for each locus as

$$
\begin{aligned}
&\tilde{m}'_1 = -e^{\pi i(M_0-M_1-2Z_1)}, \quad \tilde{m}'_2 = -e^{\pi i(M_0+M_1)}, \\
&\tilde{m}'_3 = -e^{\pi i(-M_0-M_1)}, \quad \tilde{m}'_4 = -e^{\pi i(-M_0+M_1-2Z_1)}, \\
&\tilde{t}'_1 = -e^{\pi i(M_0-M_3+2Z_3)}, \quad \tilde{t}'_2 = -e^{\pi i(-M_0-M_3)}, \\
&\tilde{t}'_3 = -e^{\pi i(M_0+M_3)}, \quad \tilde{t}'_4 = -e^{\pi i(-M_0+M_3+2Z_3)}.
\end{aligned}
\tag{20}
$$

By choosing the gauge degrees of freedom associated with the translations of x and p such that $\tilde{m}_1\tilde{m}_2\tilde{m}_3\tilde{m}_4 = \tilde{t}_1\tilde{t}_2\tilde{t}_3\tilde{t}_4 = 1$, we may write the result as

$$
\begin{aligned}
&\tilde{m}_1 = e^{\pi i(M_0-M_1-Z_1)}, \quad \tilde{m}_2 = e^{\pi i(M_0+M_1+Z_1)}, \\
&\tilde{m}_3 = e^{\pi i(-M_0-M_1+Z_1)}, \quad \tilde{m}_4 = e^{\pi i(-M_0+M_1-Z_1)}, \\
&\tilde{t}_1 = e^{\pi i(M_0-M_3+Z_3)}, \quad \tilde{t}_2 = e^{\pi i(-M_0-M_3-Z_3)}, \\
&\tilde{t}_3 = e^{\pi i(M_0+M_3-Z_3)}, \quad \tilde{t}_4 = e^{\pi i(-M_0+M_3+Z_3)}.
\end{aligned}
\tag{21}
$$

See figure 2. Note that the result (21) is consistent with the asymptotic loci for four parameter deformation with $-M_1-M_3+k = 0$ and $\pm M_0-M_3+k \geq 0$ (namely, $N_1, N_3 \geq N_2 = N_4$) obtained by constructing $\hat{\rho}_{\boldsymbol{M}}$ and its inverse explicitly [1].

Note that the moduli space of the classical curve $\rho_{\boldsymbol{M}}(x,p)^{-1} = const.$ enjoys a discrete symmetry of $W(D_5)$, the Weyl group of D_5, generated by the following transformations

$$
\begin{aligned}
&s_1 : (M_0, M_1, M_3, Z_1, Z_3) \to (M_0, M_1, Z_3, Z_1, M_3), \\
&s_2 : (M_0, M_1, M_3, Z_1, Z_3) \to (M_0, M_1, -Z_3, Z_1, -M_3), \\
&s_3 : (M_0, M_1, M_3, Z_1, Z_3) \to (M_3, M_1, M_0, Z_1, Z_3), \\
&s_4 : (M_0, M_1, M_3, Z_1, Z_3) \to (-M_1, -M_0, M_3, Z_1, Z_3), \\
&s_5 : (M_0, M_1, M_3, Z_1, Z_3) \to (M_0, Z_1, M_3, M_1, Z_3).
\end{aligned}
\tag{22}
$$

The curves at different values of $\boldsymbol{M} = (M_0, M_1, M_3, Z_1, Z_3)$ connected through $W(D_5)$ are equivalent under the coordinate transformations. Correspondingly, the quantum curve (14) $\hat{\rho}_{\boldsymbol{M}}$ is also invariant under the same $W(D_5)$ up to a similarity transformation $\hat{\rho}_{\boldsymbol{M}} \to \hat{U}\hat{\rho}_{\boldsymbol{M}}\hat{U}^{-1} = \rho_{\boldsymbol{M}}(\hat{U}\hat{x}\hat{U}^{-1}, \hat{U}\hat{p}\hat{U}^{-1})$. This implies that the normalized partition function $\frac{Z_{k,\boldsymbol{M}}(N)}{Z_{k,\boldsymbol{M}}(0)}$ is invariant under $W(D_5)$. As we have already seen above, a part of this $W(D_5)$

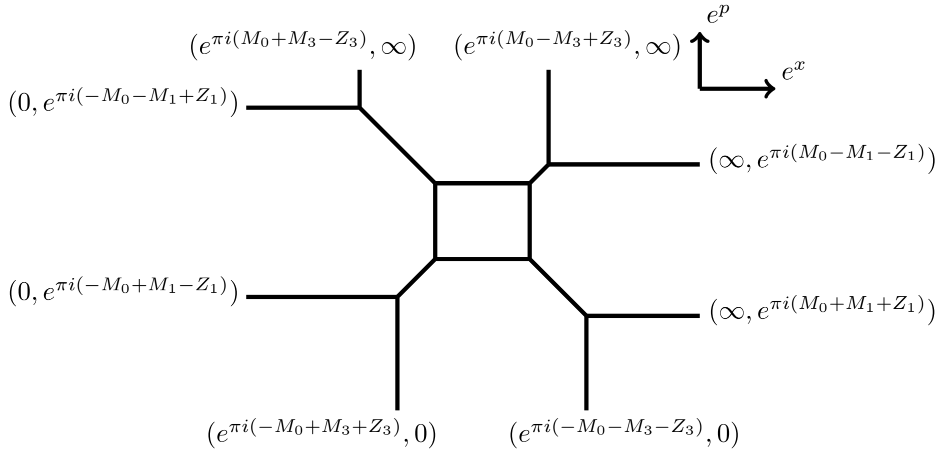


Figure 2: The asymptotic loci of the curve for the configuration (2) with generic $(M_0, M_1, M_3, Z_1, Z_3)$ obtained by the interpolation.

symmetry can be understood as the IR dualities induced by the Hanany-Witten transitions (15). However, the physical interpretations for the other elements such as the exchanges of rank differences $\{M_i\}$ and the FI parameters $\{Z_i\}$, or those which do not mix $\{M_i\}$ and $\{Z_i\}$ but still cannot be generated by the Hanany-Witten effects [19], are still not clear. It would be interesting to provide physical interpretation for these extra symmetries and also to test whether they would hold for the other observables or not.

4 Connection to 𝔮-Painlevé VI equation

In the previous section, we found that the partition function of the four-node quiver Chern-Simons theory (1) enjoys the structure of the quantization of the curve in figure 2, and hence the partition function normalized at $N = 0$ enjoys the discrete symmetry of $W(D_5)$ (22). Note that this symmetry is larger than those manifest in the type IIB brane setup (2).

Interestingly, in [1, 2] we found that the partition function, or more precisely the grand partition function $\Xi_{k,\boldsymbol{M}}(\kappa)$

$$\Xi_{k,\boldsymbol{M}}(\kappa) = \sum_{N=0}^{\infty} \kappa^N Z_{k,\boldsymbol{M}}(N) \tag{23}$$

solves the 𝔮-discrete integrable system associated with the $W(D_5)$ Weyl group called 𝔮-deformed Painlevé VI ($\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$). Although a similar connection between a matrix model and a discrete integrable system was already known for the ABJ matrix model [22] where the corresponding integrable system is 𝔮-deformed Painlevé III_3 ($\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$), our result provides a non-trivial generalization with more parameters. Below, before introducing our main results we first review the simpler connection between the ABJ matrix model and $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$. Alongside we also briefly introduce the notion of Painlevé equations and its 𝔮-discretization. After that, we display the concrete statement on the grand partition function of the four-node quiver and explain how we discovered and confirmed such statements.

4.1 Schematical structure of the equations

The partition function of the $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJ theory is given by the following $2N+M$ dimensional integration

$$Z^{\mathrm{ABJ}}_{k,M}(N) = \frac{(-1)^{MN+\frac{M(M-1)}{2}}}{N!(N+M)!}\int \frac{d^N x}{(2\pi)^N}\frac{d^{N+M}y}{(2\pi)^{N+M}} e^{\frac{ik}{4\pi}(\sum_{i=1}^N x_i^2 - \sum_{i=1}^{N+M} y_i^2)}$$
$$\times \frac{\prod_{i<j}^N (2\sinh\frac{x_i-x_j}{2})^2 \prod_{i<j}^{N+M}(2\sinh\frac{x_i-x_j}{2})^2}{\prod_{i=1}^N\prod_{j=1}^{N+M}(2\cosh\frac{x_i-y_j}{2})^2}, \tag{24}$$

which can be written in the Fermi gas formalism

$$Z^{\mathrm{ABJ}}_{k,M}(N) = \frac{Z^{\mathrm{ABJ}}_{k,M}(0)}{N!}\int \frac{d^N x}{(2\pi)^N} \det_{i,j}^{N} \langle x_i|\hat{\rho}^{\mathrm{ABJM}}_M|x_j\rangle, \tag{25}$$

where

$$Z^{\mathrm{ABJ}}_{k,M}(0) = i^{\frac{M^2}{2}-M} e^{-\frac{\pi i M(M^2-1)}{6k}} k^{-\frac{M}{2}} \prod_{r>s}^{M} 2\sin\frac{\pi(r-s)}{k} \tag{26}$$

and

$$\hat{\rho}^{\mathrm{ABJ}}_M = (-1)^M \frac{1}{2\cosh\frac{\hat{x}+\pi i M}{2}} \Big(\prod_{r=1}^M \tanh\frac{\hat{x}-t_{M,r}}{2k}\Big)\frac{1}{2\cosh\frac{\hat{p}}{2}}, \tag{27}$$

with

$$t_{n,r} = 2\pi i\Big(\frac{n+1}{2} - r\Big). \tag{28}$$

The inverse of $\hat{\rho}^{\mathrm{ABJ}}_M$ is obtained by writing (27) with quantum dilogarithm functions as follows [36]. By using quantum dilogarithm $\Phi_b(z)$

$$\Phi_b(z) = \frac{(-e^{2\pi b z + \pi i b^2}; e^{2\pi i b^2})_\infty}{(-e^{2\pi i b^{-1} z - \pi i b^{-2}}; e^{-2\pi i b^{-2}})_\infty}, \tag{29}$$

with $b=\sqrt{k}$, which satisfy the following relations

$$\frac{\Phi_b(z+ib)}{\Phi_b(z)} = \frac{1}{1+e^{\pi i b^2}e^{2\pi b z}}, \quad \frac{\Phi_b(z+ib^{-1})}{\Phi_b(z)} = \frac{1}{1+e^{-\pi i b^{-2}}e^{2\pi b^{-1} z}}, \tag{30}$$

we can express $\hat{\rho}^{\mathrm{ABJ}}_M$ as

$$\hat{\rho}^{\mathrm{ABJ}}_M = e^{\frac{\pi i M}{2}} e^{\frac{\hat{x}}{2}} \frac{\Phi(\frac{\hat{x}}{2\pi b}+\frac{ib}{2}-\frac{iM}{2b})}{\Phi(\frac{\hat{x}}{2\pi b}-\frac{ib}{2}+\frac{iM}{2b})}\frac{\Phi(\frac{\hat{x}}{2\pi b}+\frac{iM}{2b})}{\Phi(\frac{\hat{x}}{2\pi b}-\frac{iM}{2b})}\frac{1}{2\cosh\frac{\hat{p}}{2}}. \tag{31}$$

By using the first identity of quantum dilogarithm in (30), we find that the inverse of $\hat{\rho}^{\mathrm{ABJ}}_M$ is written, up to a similarity transformation which does not affect the partition functions (25), as a Laurent polynomial of $e^{\frac{\hat{x}}{2}}, e^{\frac{\hat{p}}{2}}$, which reads

$$(\hat{\rho}^{\mathrm{ABJ}}_M)^{-1} = \hat{U}^{-1}(\hat{\rho}')^{-1}\hat{U}, \quad \hat{U} = \frac{\Phi_b(\frac{\hat{x}}{2\pi b}-\frac{ib}{2}+\frac{iM}{2b})}{\Phi_b(\frac{\hat{x}}{2\pi b}+\frac{ib}{2}-\frac{iM}{2b})} e^{-\frac{\hat{x}}{2}}, \tag{32}$$

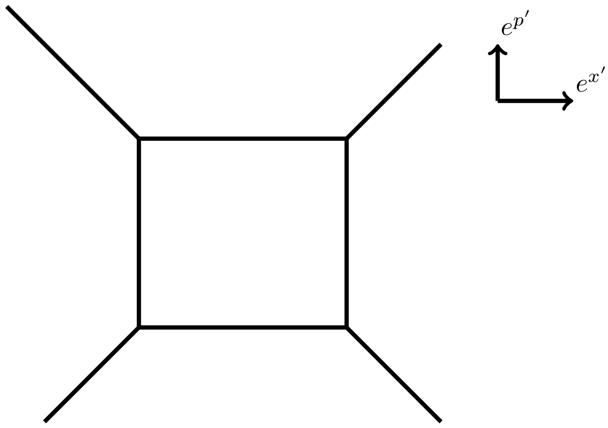


Figure 3: The classical curve $\rho'(x,p)^{-1} = const.$ corresponding to (33).

with

$$(\hat{\rho}')^{-1} = e^{-\frac{\pi i k}{4}}(e^{\hat{x}'} + e^{-\hat{p}'} + e^{\hat{p}'} + e^{\pi i(k-2M)}e^{-\hat{x}'}). \tag{33}$$

Here we have redefined the canonical position/momentum operators

$$\hat{x}' = \frac{\hat{x}+\hat{p}}{2} - \frac{3\pi i M}{2} + \frac{\pi i k}{2}, \quad \hat{p}' = \frac{-\hat{x}+\hat{p}}{2} - \frac{\pi i M}{2}, \tag{34}$$

which satisfy $[\hat{x}', \hat{p}'] = \pi i k$, to simplify the coefficients of the Laurent polynomial. The classical limit of (33) gives a genus-one curve with four asymptotic points, as depicted in figure 3.

Fermi gas formalism is also an efficient tool to calculate the exact values of $Z_{k,M}(N)$ for finite N. In [20, 22] it was observed that these are infinitely many non-linear relations among the exact values $Z_{k,M}(N)$ at different values of M, N, which are summarized in the following form

$$\Xi^{\text{ABJ}}_{k,M+1}(\kappa)\Xi^{\text{ABJ}}_{k,M-1}(\kappa) = \Xi^{\text{ABJ}}_{k,M}(\kappa)^2 + e^{\pi i(1-\frac{2M}{k})}\Xi^{\text{ABJ}}_{k,M}(-\kappa)^2, \tag{35}$$

where $\Xi^{\text{ABJ}}_{k,M}(\kappa)$ is the grand partition function of the ABJ theory with respect to the overall rank N

$$\Xi^{\text{ABJ}}_{k,M}(\kappa) = \sum_{N=0}^{\infty} \kappa^N Z^{\text{ABJ}}_{k,M}(N). \tag{36}$$

The bilinear equation (35) is known as the Hirota bilinear (τ-) form of $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$ (or the $\mathfrak{q}$-deformed affine SU(2) Toda equation) [37–39]

$$\tau_i(qt)\tau_i(q^{-1}t) = \tau_i(t)^2 + t^{\frac{1}{2}}\tau_j(t)^2, \quad (i = 1, 2,\ j \neq i), \tag{37}$$

with the following parameter identification

$$\mathfrak{q} = e^{\frac{4\pi i}{k}}, \quad t = e^{2\pi i(1-\frac{2M}{k})}, \quad \tau_1(t) = \Xi^{\text{ABJ}}_{k,M}(\kappa), \quad \tau_2(t) = \Xi^{\text{ABJ}}_{k,M}(-\kappa). \tag{38}$$

Painlevé equations were originally considered under the motivation to define new special functions as solutions of non-linear ordinary differential equations. Partly based on the insight from the classical integrable systems, Painlevé, et.al. restricted themselves to the differential equations whose solution does not have a movable (i.e. depending on the initial conditions) branch point. As a result, they found that the second order ordinary differential equations with this property and whose general solution cannot be written with hypergeometric or elliptic functions are classified into six types called $\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{III}}$, $\mathrm{P_{IV}}$, $\mathrm{P_V}$ and $\mathrm{P_{VI}}$. $\mathrm{P_{III_3}}$ is a specialization of $\mathrm{P_{III}}$ whose explicit expression is given as

$$\mathrm{P_{III_3}}: \quad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\Big(\frac{d\lambda}{dt}\Big)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{2\lambda^2}{t^2} - \frac{2}{t}. \tag{39}$$

By introducing τ-functions τ_1, τ_2 as

$$\frac{d^2(\log\tau_1)}{d(\log t)^2} = \lambda, \quad \frac{d^2(\log\tau_2)}{d(\log t)^2} = t\lambda^{-1}, \tag{40}$$

$\mathrm{P_{III_3}}$ is written in the Hirota bilinear form

$$\tau_i\frac{d^2\tau_i}{d(\log t)^2} - \Big(\frac{d\tau_i}{d(\log t)}\Big)^2 = t^{\frac{1}{2}}\tau_j^2, \quad (i=1,2,\ j\neq i). \tag{41}$$

If we further "$\mathfrak{q}$-discretize" the derivative as $\partial_{\log t} f \to \frac{f(\mathfrak{q}t)-f(t)}{\mathfrak{q}-1}$, (41) is uplifted to the difference equation (37).

($\mathfrak{q}$-)Painlevé equations appears in various problems in physics. In particular, it is known that $\mathfrak{q}\mathrm{P_{III_3}}$ in τ-form is also satisfied by the discrete Fourier transform of the Nekrasov partition function (called Nekrasov-Okounkov partition function Z_{NO}) of the five dimensional $\mathcal{N}=1$ SU(2) pure Yang-Mills theory [37]. Interestingly, the relation between the parameters of $\mathfrak{q}\mathrm{P_{III_3}}$ and the 5d parameters precisely coincides with the one read off from $\rho'(x,p)^{-1}$ by regarding it as the five-dimensional Seiberg-Witten curve. The fact that the grand partition function of the ABJ theory and the Nekrasov-Okounkov partition function obeys the same $\mathfrak{q}$-difference equation is consistent with the topological string/spectral theory (TS/ST) correspondence [18]. Indeed, TS/ST correspondence claims[4] that the Fredholm determinant of a trace class operator $\rho(\hat{x},\hat{p})$ is given by the discrete Fourier transform of a non-perturbative completion [42] of the topological string partition function on a non-compact Calabi-Yau threefold whose mirror curve is $\rho(x,p)^{-1} = const.$, which is further related, via geometric engineering [43], to the Nekrasov partition function of five-dimensional theory whose Seiberg-Witten curve is $\rho(x,p)^{-1} = const.$. The correspondence between the $\mathfrak{q}$-Painlevé τ-function and the Nekrasov-Okounkov partition function of five-dimensional $\mathcal{N}=1$ $SU(2)$ Yang-Mills theory is known as the $\mathfrak{q}$-uplift of the Painlevé/gauge correspondence [44]. Similar connections exist also for the other $\mathfrak{q}$-Painlevé equations, where the Seiberg-Witten curve of the five-dimensional theory (or its discrete symmetry) is identified with those in the Sakai's classification of the $\mathfrak{q}$-Painlevé equations by surface/symmetry types [21, 23].

From the viewpoint of Painlevé/gauge correspondence, $\mathfrak{q}\mathrm{P_{VI}}$ is characterized as a set of bilinear equations satisfied by the Nekrasov-Okounkov partition function of 5d $\mathcal{N}=1$ $SU(2)$ Yang-Mills theory with $N_{\mathrm{f}}=4$ [45]

$$Z_{\mathrm{NO}}^{\mathrm{SU(2)},N_{\mathrm{f}}=4}(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma,s;t) = \sum_{n\in\mathbb{Z}} s^n t^{(\sigma+n)^2-\theta_t^2-\theta_0^2} C(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma+n) Z(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma+n,t) \tag{42}$$

[4]Although there are still no rigorous proofs for the TS/ST correspondence, the claim is confirmed by various non-trivial pieces of evidences [17, 27–29, 40, 41].

with

$$C(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma)=\frac{\prod_{\epsilon,\epsilon'=\pm}G_{\mathfrak{q}}(1+\epsilon\theta_\infty-\theta_1+\epsilon'\sigma)G_{\mathfrak{q}}(1+\epsilon\sigma-\theta_t+\epsilon'\theta_0)}{G_{\mathfrak{q}}(1+2\sigma)G_{\mathfrak{q}}(1-2\sigma)},$$

$$Z(\theta_0,\theta_1,\theta_t,\theta_\infty;\sigma,t)=\sum_{\lambda_+,\lambda_-}\frac{\prod_{\epsilon,\epsilon'=\pm}N_{\phi,\lambda_{\epsilon'}}(\mathfrak{q}^{\epsilon\theta_\infty-\theta_1-\epsilon'\sigma})N_{\lambda_\epsilon,\phi}(\mathfrak{q}^{\epsilon\sigma-\theta_t-\epsilon'\theta_0})}{\prod_{\epsilon,\epsilon'=\pm}N_{\lambda_\epsilon,\lambda_{\epsilon'}}(\mathfrak{q}^{(\epsilon-\epsilon')\sigma})},\tag{43}$$

as

$$\tau_1(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1,\theta_t,\theta_\infty+\frac{1}{2};\sigma,s;t\Big),$$
$$\tau_2(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1,\theta_t,\theta_\infty-\frac{1}{2};\sigma,s;t\Big),$$
$$\tau_3(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0+\frac{1}{2},\theta_1,\theta_t,\theta_\infty;\sigma+\frac{1}{2},s;t\Big),$$
$$\tau_4(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0-\frac{1}{2},\theta_1,\theta_t,\theta_\infty;\sigma-\frac{1}{2},s;t\Big),$$
$$\tau_5(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1-\frac{1}{2},\theta_t,\theta_\infty;\sigma,s;t\Big),$$
$$\tau_6(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1+\frac{1}{2},\theta_t,\theta_\infty;\sigma,s;t\Big),$$
$$\tau_7(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1,\theta_t-\frac{1}{2},\theta_\infty;\sigma+\frac{1}{2},s;t\Big),$$
$$\tau_8(t)=Z_{\mathrm{NO}}^{\mathrm{SU}(2),N_{\mathrm{f}}=4}\Big(\theta_0,\theta_1,\theta_t+\frac{1}{2},\theta_\infty;\sigma-\frac{1}{2},s;t\Big),$$
$$\tau_1(t)\tau_2(t)-t\mathfrak{q}^{-2\theta_1}\tau_3(t)\tau_4(t)-(1-t\mathfrak{q}^{-2\theta_1})\tau_5(t)\tau_6(t)=0,\tag{44}$$
$$\tau_1(t)\tau_2(t)-t\tau_3(t)\tau_4(t)-(1-t\mathfrak{q}^{-2\theta_t})\tau_5(\mathfrak{q}^{-1}t)\tau_6(\mathfrak{q}t)=0,\tag{45}$$
$$\tau_1(t)\tau_2(t)-\tau_3(t)\tau_4(t)+(1-t\mathfrak{q}^{-2\theta_1})\mathfrak{q}^{2\theta_t}\tau_7(\mathfrak{q}^{-1}t)\tau_8(\mathfrak{q}t)=0,\tag{46}$$
$$\tau_1(t)\tau_2(t)-\mathfrak{q}^{2\theta_t}\tau_3(t)\tau_4(t)+(1-t\mathfrak{q}^{-2\theta_t})\mathfrak{q}^{2\theta_t}\tau_7(t)\tau_8(t)=0,\tag{47}$$
$$\tau_5(\mathfrak{q}^{-1}t)\tau_6(t)+t\mathfrak{q}^{-\theta_1-\theta_\infty+\theta_t-\frac{1}{2}}\tau_7(\mathfrak{q}^{-1}t)\tau_8(t)-\tau_1(\mathfrak{q}^{-1}t)\tau_2(t)=0,\tag{48}$$
$$\tau_5(\mathfrak{q}^{-1}t)\tau_6(t)+t\mathfrak{q}^{-\theta_1+\theta_\infty+\theta_t-\frac{1}{2}}\tau_7(\mathfrak{q}^{-1}t)\tau_8(t)-\tau_1(t)\tau_2(\mathfrak{q}^{-1}t)=0,\tag{49}$$
$$\tau_5(\mathfrak{q}^{-1}t)\tau_6(t)+t\mathfrak{q}^{\theta_0+2\theta_t}\tau_7(\mathfrak{q}^{-1}t)\tau_8(t)-\mathfrak{q}^{\theta_t}\tau_3(\mathfrak{q}^{-1}t)\tau_4(t)=0,\tag{50}$$
$$\tau_5(\mathfrak{q}^{-1}t)\tau_6(t)+t\mathfrak{q}^{-\theta_0+2\theta_t}\tau_7(\mathfrak{q}^{-1}t)\tau_8(t)-\mathfrak{q}^{\theta_t}\tau_3(t)\tau_4(\mathfrak{q}^{-1}t)=0.\tag{51}$$

Here $G_{\mathfrak{q}}(x)$ is the $\mathfrak{q}$-deformed Barnes G-function characterized by the second order recursion relation $\frac{G_{\mathfrak{q}}(x+1)G_{\mathfrak{q}}(x-1)}{G_{\mathfrak{q}}(x)^2}=\frac{1-\mathfrak{q}^{x-1}}{1-\mathfrak{q}}$ and $N_{\mu,\lambda}(x)$ is the five-dimensional Nekrasov factor which is defined as

$$N_{\mu,\lambda}(x)=\prod_{\square\in\lambda}(1-\mathfrak{q}^{-\ell_\lambda(\square)-a_\mu(\square)-1}x)\prod_{\square\in\mu}(1-\mathfrak{q}^{a_\lambda(\square)+\ell_\mu(\square)+1}x)\tag{52}$$

with $a_\lambda(\square)$ and $\ell_\lambda(\square)$ being the arm length and leg length at point $\square=(i,j)$ for a Young diagram $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_d)$ in the Frobenius notation (see figure 4 for example)

$$a_\lambda(\square)=\begin{cases}\lambda_i-j & (i\le d)\\ -j & (i>d)\end{cases},\quad \ell_\lambda(\square)=\begin{cases}\lambda'_j-i & (j\le d')\\ -i & (j>d')\end{cases},\quad {}^t\lambda=(\lambda'_1,\lambda'_2,\cdots,\lambda'_{d'})\text{: transpose of }\lambda.\tag{53}$$

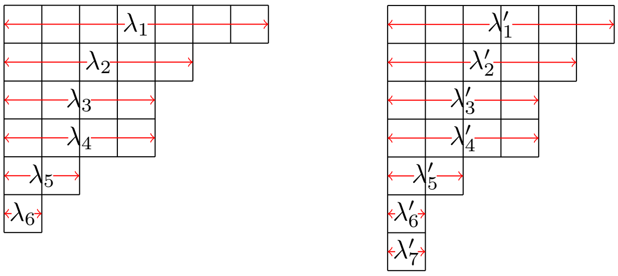


Figure 4: An example of a pair of Yound diagram $\lambda = (\lambda_1, \cdots, \lambda_d)$ and its transpose ${}^t\lambda = (\lambda'_1, \cdots, \lambda'_{d'})$. In the figure $d = 6$, $\lambda = (7, 5, 4, 4, 2, 1)$, $d' = 7$ and ${}^t\lambda = (6, 5, 4, 4, 2, 1, 1)$.

See [1,45] for detail. Indeed, this theory has global symmetry of D_5 which is an enhancement from the SU(4) flavor symmetry due to the instantons [46,47]. More concretely, the Seiberg-Witten curve of this theory

$$q\Big(\frac{m_1m_2}{m_3m_4}\Big)^{\frac{1}{2}}v^{-1}w-((m_1m_2)^{\frac{1}{2}}+q(m_3m_4)^{-\frac{1}{2}})w+vw-q\Big(\frac{m_1m_2}{m_3m_4}\Big)^{\frac{1}{2}}(m_3+m_4)v^{-1}+E-(m_1+m_2)v$$
$$+q(m_1m_2m_3m_4)^{\frac{1}{2}}v^{-1}w^{-1}-m_1m_2((m_1m_2)^{-\frac{1}{2}}+q(m_3m_4)^{\frac{1}{2}})w^{-1}+m_1m_2vw^{-1}=0, \tag{54}$$

where the coefficients are related to $\theta_0, \theta_1, \theta_t, \theta_\infty$ and t as

$$\theta_0=\frac{1}{4\pi i}\log\frac{m_1}{m_3},\quad \theta_1=\frac{1}{4\pi i}\log(m_2m_4),\quad \theta_t=\frac{1}{4\pi i}\log\frac{1}{m_1m_3},$$
$$\theta_\infty=\frac{1}{4\pi i}\log\frac{m_4}{m_2},\quad \frac{\log t}{\log \mathfrak{q}}=\frac{1}{4\pi i}\log\frac{q^2m_2m_4}{m_1m_3}, \tag{55}$$

is of the same form as the quantum curve of the Fermi gas formalism $\hat{\rho}_{\boldsymbol{M}}^{-1} = \sum_{m,n=-1,0,1} c_{mn} e^{m\hat{x}+n\hat{p}}$ of the four-node quiver super Chern-Simons theory (1) with $(v, w) = (e^x, e^p)$.

Now let us conjecture that the Fredholm determinant of the $(2, 2)$ model gives the τ-function of $\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$,

$$\tau^{\mathrm{3d}}_{k,\boldsymbol{M}} \sim \mathrm{Det}(1+\kappa\hat{\rho}_{\boldsymbol{M}}). \tag{56}$$

From the curves, we can read off the dictionary between the five-dimensional parameters $(\theta_0, \theta_1, \theta_t, \theta_\infty, t)$ and the three-dimensional parameters $\boldsymbol{M} = (M_0, M_1, M_3, Z_1, Z_3)$ as

$$\theta_0=\frac{M_0-Z_1}{2},\quad \theta_1=\frac{M_1+Z_3}{2},\quad \theta_t=\frac{M_1-Z_3}{2},$$
$$\theta_\infty=\frac{-M_0-Z_1}{2},\quad t=\mathfrak{t}^{M_1+M_3}=e^{\frac{2\pi i(M_1+M_3)}{k}}, \tag{57}$$

where we have identified $\mathfrak{q}$ with the Planck constant of the Fermi gas formalism of the four-node quiver as $\mathfrak{q} = e^{\frac{4\pi^2 i}{\hbar}} = e^{\frac{2\pi i}{k}}$, by consulting the result for ABJ theory (38). Then we can identify how $\boldsymbol{M}$ should be shifted corresponding to each of the tau functions in the bilinear equations (44)-(51). In this way, we can write down the schematic form of the bilinear equations which $\tau^{\rm 3d}_{k,\boldsymbol{M}}$ should satisfy, as

$$
\begin{aligned}
(44)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,-\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,-\frac{1}{2},\frac{1}{2},0,-\frac{1}{2})}&=0,\\
(45)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,-\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},\frac{1}{2},0,\frac{1}{2})}&=0,\\
(46)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,-\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2})}&=0,\\
(47)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,0,-\frac{1}{2},0)}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},-\frac{1}{2},0,-\frac{1}{2})}&=0,\\
(48)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,-\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,\frac{1}{2},-\frac{1}{2},0)}&=0,\\
(49)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,-\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(-\frac{1}{2},0,\frac{1}{2},-\frac{1}{2},0)}&=0,\\
(50)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,-\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2},0)}&=0,\\
(51)\to\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(0,\frac{1}{2},0,0,-\frac{1}{2})}+\bigcirc\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}\pm(-\frac{1}{2},0,-\frac{1}{2},\frac{1}{2},0)}&=0.
\end{aligned}
\tag{58}
$$

We see that the shifts in three terms in each equation are of the following form: (i) two coordinates of $\boldsymbol{M}$ shifted by $(\pm 1/2,\pm 1/2)$, (ii) the same two coordinates shifted by $(\pm 1/2,\mp 1/2)$ and (iii) the other three coordinates shifted by $(\pm\sigma/2,\pm\sigma'/2,\pm\sigma''/2)$ with $(\sigma,\sigma',\sigma'')$ one of $\{(+++),(+--),(-+-),(--+)\}$. Since $\mathrm{Det}(1+\kappa\hat{\rho}_{\boldsymbol{M}})$ is invariant under $W(E_5)$ which is generated by the simple reflections (22) it is natural to expect that the same equations exist for all choices for the two components and the three signs, namely ${}_4C_2\times 4=40$ equations in total. After all, our conjecture is that the Fredholm determinant satisfies the following equations [2]:

$$
\begin{aligned}
&\tau^{\rm 3d}_{k,\boldsymbol{M}}(\kappa)=F_{k,\boldsymbol{M}}\mathrm{Det}(1+\kappa\hat{\rho}_{\boldsymbol{M}}),\\
&f_1^{(a,b,\sigma_c,\sigma_d,\sigma_e)}\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}_\alpha\pm\frac{1}{2}(\delta^a_\alpha+\delta^b_\alpha)}((\gamma_1^{(a,b,\sigma_c,\sigma_d,\sigma_e)})^{\pm 1}\kappa)\\
&\quad+f_2^{(a,b,\sigma_c,\sigma_d,\sigma_e)}\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}_\alpha\pm\frac{1}{2}(\delta^a_\alpha-\delta^b_\alpha)}((\gamma_2^{(a,b,\sigma_c,\sigma_d,\sigma_e)})^{\pm 1}\kappa)\\
&\qquad+f_3^{(a,b,\sigma_c,\sigma_d,\sigma_e)}\prod_{\pm}\tau^{\rm 3d}_{k,\boldsymbol{M}_\alpha\pm\frac{1}{2}(\sigma_c\delta^c_\alpha+\sigma_d\delta^d_\alpha+\sigma_e\delta^e_\alpha)}((\gamma_3^{(a,b,\sigma_c,\sigma_d,\sigma_e)})^{\pm 1}\kappa)=0.
\end{aligned}
\tag{59}
$$

Here we have introduced the overall factors F, f_1, f_2, f_3 and the parameters $\gamma_1,\gamma_2,\gamma_3$ associated with the relation between κ and σ as unknown parameters. We would like to fix these parameters by hand so that the bilinear relations are indeed satisfied.

4.2 Fixing coefficients $F, f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3$

Since the bilinear relations (59) do not depends explicitly on κ, let us first consider the case $\kappa = 0$ and determine f_1, f_2, f_3 and F from the following bilinear relations

$$f_1^{(a,b,\sigma_1,\sigma_2,\sigma_3)} \prod_{\pm} F_{k,\boldsymbol{M}_\alpha \pm \frac{1}{2}(\delta_\alpha^a + \delta_\alpha^b)} + f_2^{(a,b,\sigma_1,\sigma_2,\sigma_3)} \prod_{\pm} F_{k,\boldsymbol{M}_\alpha \pm \frac{1}{2}(\delta_\alpha^a - \delta_\alpha^b)} + f_3^{(a,b,\sigma_1,\sigma_2,\sigma_3)} \prod_{\pm} F_{k,\boldsymbol{M}_\alpha \pm \frac{1}{2}(\sigma_1\delta_\alpha^c + \sigma_2\delta_\alpha^d + \sigma_3\delta_\alpha^e)} = 0. \tag{60}$$

Note that a part of the information of f_1, f_2, f_3 can be absorbed into F, hence in the mathematics context the coefficients of the bilinear relations are not regarded essential in arguing the properties of $\mathfrak{q}$-Painlevé equations [48]. Hence a reasonable approach is not to guess f_1, f_2, f_3, F at the same time, but rather to fix F by hand first and then ask what f_1, f_2, f_3 are. From the three-dimensional viewpoint there is a natural choice for F:

$$F_{k,\boldsymbol{M}} = Z_{k,\boldsymbol{M}}(N = 0), \tag{61}$$

with which $\tau^{3\text{d}}_{k,\boldsymbol{M}}$ is the grand partition function

$$\tau^{3\text{d}}_{k,\boldsymbol{M}} = \Xi_{k,\boldsymbol{M}}(\kappa) = \sum_{N=0}^{\infty} \kappa^N Z_{k,\boldsymbol{M}}(N). \tag{62}$$

Indeed, in the case of the ABJ theory the partition function at $N = 0$, $Z^{\text{ABJ}}_{k,M}(0)$ (26), satisfies the second order $\mathfrak{q}$-difference relation

$$Z^{\text{ABJ}}_{k,M+1}(0) Z^{\text{ABJ}}_{k,M-1}(0) = (1 - e^{-\frac{2\pi i M}{k}}) Z^{\text{ABJ}}_{k,M}(0)^2, \tag{63}$$

which is the same as $\mathfrak{q}\text{P}_{\text{III}_3}$ (35). Hence we may expect that also in the current case $F_{k,\boldsymbol{M}} = Z_{k,\boldsymbol{M}}(N = 0)$ will satisfy the 40 bilinear relations with relatively simple coefficients f_1, f_2, f_3.

In [1] it was found that the partition function of the four-node quiver super Chern-Simons theory (4) can be written in the following form

$$Z_{k,\boldsymbol{M}}(N) = e^{i\Theta_{k,\boldsymbol{M}}} \frac{Z_k^{\text{CS}}(L_1) Z_k^{\text{CS}}(L_2)}{N!} \int \prod_{i=1}^{N} \frac{dx_i}{2\pi} \det \begin{pmatrix} [\langle x_i|\widehat{D}_1\widehat{D}_2|x_j\rangle]_{(i,j)} & [\langle x_i|\widehat{D}_1\widehat{d}_2|-t_{L,s}\rangle\!\rangle]_{(i,s)} \\ [\langle\!\langle t_{L,r}|\widehat{d}_1\widehat{D}_2|x_j\rangle]_{(r,j)} & [\langle\!\langle t_{L,r}|\widehat{d}_1\widehat{d}_2|-t_{L,s}\rangle\!\rangle]_{(r,s)} \end{pmatrix}, \tag{64}$$

where

$$L_1 = -M_0 - M_3 + k, \quad L_2 = M_0 - M_3 + k, \quad L = -M_1 - M_3 + k, \tag{65}$$

$t_{n,r}$ is defined as (28), and

$$\Theta_{k,\boldsymbol{M}} = \frac{\pi}{k}(M_0 - M_1 - M_3) Z_1 Z_3 - \pi Z_1 Z_3 - 2\pi M_1 (Z_1 + Z_3),$$
$$Z_k^{\text{CS}}(n) = \frac{1}{k^{\frac{n}{2}}} \prod_{j<j'}^{n} 2 \sin \frac{\pi(j' - j)}{k}. \tag{66}$$

The matrix elements of the $(N+L) \times (N+L)$ matrix in the determinant in (64) are written in the notation of one-dimensional quantum mechanics (7) with the operators $\hat{D}_1, \hat{D}_2, \hat{d}_1, \hat{d}_2$

give as

$$\hat{D}_1 = e^{\frac{Z_1}{k}\hat{x}} S_{L_1}(\hat{x}) \frac{1}{2\cosh\frac{\hat{p}-\pi iL}{2}} e^{-\frac{Z_1}{k}\hat{x}} C_{L_1}(\hat{x}), \quad \hat{d}_1 = e^{-\frac{Z_1}{k}\hat{x}} C_{L_1}(\hat{x}),$$
$$\hat{D}_2 = C_{L_2}(\hat{x}-2\pi iZ_3) \frac{1}{2\cosh\frac{\hat{p}+\pi iL}{2}} S_{L_2}(\hat{x}-2\pi iZ_3), \quad \hat{d}_2 = C_{L_2}(\hat{x}-2\pi iZ_3), \tag{67}$$

where

$$S_n(x) = i^n \frac{\prod_{r=1}^n 2\sinh\frac{x-t_{n,r}}{2k}}{2\cosh\frac{x+\pi in}{2}}, \quad C_n(x) = \frac{1}{\prod_{r=1}^n 2\cosh\frac{x-t_{n,r}}{2k}}. \tag{68}$$

In particular, the partition function at $N=0$ for various k, M_0, M_1, M_3 is given in a closed form as function of Z_1, Z_3 as

$$Z_{k,\boldsymbol{M}}(0) = e^{i\Theta_{k,\boldsymbol{M}}} Z_k^{\rm CS}(L_1) Z_k^{\rm CS}(L_2) \det\left([\langle\langle t_{L,r}|\hat{d}_1^{\rm VI}\hat{d}_2^{\rm VI}|-t_{L,s}\rangle\rangle]_{(r,s)}\right)$$
$$= e^{i\Theta_{k,\boldsymbol{M}}} Z_k^{\rm CS}(L_1) Z_k^{\rm CS}(L_2) \det\left([I_{k,L_1+L_2}(L+1-r-s-Z_1, \{t_{L_1,r'}\}_{r'=1}^{L_1} \cup \{2\pi iZ_3 + t_{L_2,r'}\}_{r'=1}^{L_2})]_{(r,s)}\right), \tag{69}$$

where $I_{k,n}(\alpha,\{\beta_a\})$ denotes the following integration[5]

$$I_{k,n}(\alpha,\{\beta_a\}) = \int_{-\infty}^{\infty} \frac{dx}{2\pi k} \frac{e^{\frac{\alpha x}{k}}}{\prod_{a=1}^n 2\cosh\frac{x-\beta_a}{2k}} = \frac{1}{e^{-\pi i\alpha}-(-1)^n e^{\pi i\alpha}} \sum_{a=1}^n \frac{e^{\frac{\alpha\beta_a}{k}}}{\prod_{a'(\neq a)} 2i\sinh\frac{\beta_a-\beta_{a'}}{2k}}. \tag{70}$$

By using these results, we indeed find that $Z_{k,\boldsymbol{M}}(0)$ satisfies the following 40 bilinear relations:

$$e^{-\frac{\pi i}{2k}(\sigma_c c+\sigma_d d+\sigma_e e)} S_{\boldsymbol{M}}^{(1)} \prod_\pm Z_{k,\boldsymbol{M}\pm\frac{1}{2}(\delta_\alpha^a+\delta_\alpha^b)}(0) + e^{\frac{\pi i}{2k}(\sigma_c c+\sigma_d d+\sigma_e e)} S_{\boldsymbol{M}}^{(1)} \prod_\pm Z_{k,\boldsymbol{M}\pm\frac{1}{2}(\delta_\alpha^a-\delta_\alpha^b)}(0)$$
$$+ S_{\boldsymbol{M}}^{(3)} \prod_\pm Z_{k,\boldsymbol{M}\pm\frac{1}{2}(\sigma_c\delta_\alpha^c+\sigma_d\delta_\alpha^d+\sigma_e\delta_\alpha^e)}(0) = 0, \tag{71}$$

where $S_{\boldsymbol{M}}^{(1)}, S_{\boldsymbol{M}}^{(2)}$ and $S_{\boldsymbol{M}}^{(3)}$ for each $(a,b;\sigma_c,\sigma_d,\sigma_e)$ are

$$(a,b) = (M_1,Z_1) \to S_{\boldsymbol{M}}^{(1)} = S_1^+, \quad S_{\boldsymbol{M}}^{(2)} = S_1^-, \quad S_{\boldsymbol{M}}^{(3)} = S_3^{\sigma_{M_0}},$$
$$(a,b) = (M_3,Z_3) \to S_{\boldsymbol{M}}^{(1)} = S_3^+, \quad S_{\boldsymbol{M}}^{(2)} = S_3^-, \quad S_{\boldsymbol{M}}^{(3)} = S_1^{\sigma_{M_0}},$$
$$(a,b) = (M_0,M_1) \to S_{\boldsymbol{M}}^{(1)} = S_{\boldsymbol{M}}^{(2)} = 1, \quad S_{\boldsymbol{M}}^{(3)} = S_3^{\sigma_{Z_1}},$$
$$(a,b) = (M_0,M_3) \to S_{\boldsymbol{M}}^{(1)} = S_{\boldsymbol{M}}^{(2)} = 1, \quad S_{\boldsymbol{M}}^{(3)} = S_1^{\sigma_{Z_3}},$$
$$(a,b) = (M_0,Z_1) \to S_{\boldsymbol{M}}^{(1)} = S_{\boldsymbol{M}}^{(2)} = 1, \quad S_{\boldsymbol{M}}^{(3)} = S_3^{\sigma_{M_1}},$$
$$(a,b) = (M_0,Z_3) \to S_{\boldsymbol{M}}^{(1)} = S_{\boldsymbol{M}}^{(2)} = 1, \quad S_{\boldsymbol{M}}^{(3)} = S_1^{\sigma_{M_3}},$$
$$(a,b)\text{: others} \to S_{\boldsymbol{M}}^{(1)} = S_{\boldsymbol{M}}^{(2)} = S_{\boldsymbol{M}}^{(3)} = 1, \tag{72}$$

with

$$S_{1/3}^\pm = 2\sinh\frac{\pi(M_{1/3}\pm Z_{1/3})}{k}. \tag{73}$$

[5]Here we ignore the issue of convergence of the integration in $I_{k,n}$ (70) as well as the convergence of the original integration (64), which we address more carefully in a coming paper [35].

The expression (64) also allows us to calculate $Z_{k,\boldsymbol{M}}(N)$ for $N\ge 0$ for various values of k,M_0,M_1,M_3 and as a function of Z_1,Z_3. For the details of the calculation, see [2]. With these exact values, we find that the correct choice of $\gamma_1,\gamma_2,\gamma_3$ with which $q\mathrm{P}_{\mathrm{VI}}$ bilinear relations are satisfied is very simple:

$$\gamma_1=1,\quad \gamma_2=-1,\quad \gamma_3=-i. \tag{74}$$

Namely, $\gamma_1,\gamma_2,\gamma_3$ are the same for all 40 bilinear relations and they do not depend on $k,\boldsymbol{M}$ either. Indeed, since the Fredholm determinant $\mathrm{Det}(1+\kappa\hat{\rho}_{\boldsymbol{M}})$ is invariant under $W(D_5)$, we can generate several equations for the same choice of the shift directions $(a,b;\sigma_c,\sigma_d,\sigma_e)$ as the Weyl orbit of the other equations. In order for these equations to be satisfied at all order in κ, the $\boldsymbol{M}$-dependence of γ's is highly constrained and it is rather unlikely for γ's to have a non-trivial dependence on $\boldsymbol{M}$. In summary, we conjecture that the grand partition function (62) satisfies the following bilinear relations

$$\begin{aligned} e^{-\frac{\pi i}{2k}(\sigma_c c+\sigma_d d+\sigma_e e)}S^{(1)}_{\boldsymbol{M}}\prod_{\pm}\Xi_{k,\boldsymbol{M}\pm\frac{1}{2}(\delta^a_\alpha+\delta^b_\alpha)}(\kappa)+e^{\frac{\pi i}{2k}(\sigma_c c+\sigma_d d+\sigma_e e)}S^{(1)}_{\boldsymbol{M}}\prod_{\pm}\Xi_{k,\boldsymbol{M}\pm\frac{1}{2}(\delta^a_\alpha-\delta^b_\alpha)}(-\kappa)\\ +S^{(3)}_{\boldsymbol{M}}\prod_{\pm}\Xi_{k,\boldsymbol{M}\pm\frac{1}{2}(\sigma_c\delta^c_\alpha+\sigma_d\delta^d_\alpha+\sigma_e\delta^e_\alpha)}(\mp i\kappa)=0. \end{aligned}\tag{75}$$

Here let us display one of the simplest non-trivial examples to see that (75) is actually satisfied. For simplicity let us consider the bilinear relation labeled by $(Z_1,Z_3;\sigma_{M_0}=-1,\sigma_{M_1}=+1,\sigma_{M_3}=-1)$ at $(M_0,M_1,M_3)=(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2})$. The bilinear equation involves the grand partition function evaluated at the following six points:

$$\begin{aligned} &\Big(M_0=-\frac{1}{2},M_1=\frac{1}{2},M_3=k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\pm\frac{1}{2}\Big)\to(L_1,L_2,L)=(1,0,0),\\ &\Big(M_0=-\frac{1}{2},M_1=\frac{1}{2},M_3=k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\mp\frac{1}{2}\Big)\to(L_1,L_2,L)=(1,0,0),\\ &\Big(M_0=-1,1,k-1,Z_1,Z_3\Big)\to(L_1,L_2,L)=(2,0,0),\\ &\Big(M_0=0,0,k,Z_1,Z_3\Big)\to(L_1,L_2,L)=(0,0,0). \end{aligned}\tag{76}$$

Here we have written the rank variables both in terms of M_0,M_1,M_3 and in terms of L_1,L_2,L to visualize the simplicity of this choice for the purpose of calculating exact values of $Z_{k,\boldsymbol{M}}(N)$ from (64). At order κ^0 the bilinear relation reduces to (71), which is

$$\begin{aligned} &ie^{-\frac{3\pi i}{4k}}\prod_{\pm}Z_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\pm\frac{1}{2})}(0)-ie^{\frac{3\pi i}{4k}}\prod_{\pm}Z_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\mp\frac{1}{2})}(0)\\ &+Z_{k,(-1,1,k-1,Z_1,Z_3)}(0)Z_{k,(0,0,k,Z_1,Z_3)}(0)=0 \end{aligned}\tag{77}$$

for $(a,b;\sigma_c,\sigma_d,\sigma_e)=(Z_1,Z_3;\sigma_{M_0}=-1,\sigma_{M_1}=+1,\sigma_{M_3}=-1)$ at $(M_0,M_1,M_3)=(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2})$. From (69) we can calculate the relevant $Z_{k,\boldsymbol{M}}(0)$ as

$$\begin{aligned} &Z_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1,Z_3)}(0)=e^{\pi i[(-\frac{1}{2k}-2)Z_1Z_3-(Z_1+Z_3)]}k^{-\frac{1}{2}},\\ &Z_{k,(-1,1,k-1,Z_1,Z_3)}(0)=e^{\pi i[(-\frac{1}{k}-2)Z_1Z_3-2(Z_1+Z_3)]}k^{-1}\Big(2\sin\frac{\pi}{k}\Big),\\ &Z_{k,(0,0,k,Z_1,Z_3)}(0)=e^{-2\pi iZ_1Z_3}. \end{aligned}\tag{78}$$

Substituting these into the left-hand side of (77) we find that (77) is indeed satisfied. To test the bilinear relation at higher order in κ it is useful to write them in terms of the normalized grand partition function

$$\Xi^{\text{norm}}_{k,\boldsymbol{M}}(\kappa)=\frac{\Xi_{k,\boldsymbol{M}}(\kappa)}{Z_{k,\boldsymbol{M}}(0)}=1+\sum_{N=1}^{\infty}\kappa^N\frac{Z_{k,\boldsymbol{M}}(N)}{Z_{k,\boldsymbol{M}}(0)}. \tag{79}$$

After substituting (78), the coefficients of the bilinear relation (75) for $(a,b;\sigma_c,\sigma_d,\sigma_e)=(Z_1,Z_3;\sigma_{M_0}=-1,\sigma_{M_1}=+1,\sigma_{M_3}=-1)$ at $(M_0,M_1,M_3)=(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2})$ reduce as

$$\begin{aligned}&-ie^{-\frac{\pi i}{k}}\prod_{\pm}\Xi^{\text{norm}}_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\pm\frac{1}{2})}(\kappa)+ie^{\frac{\pi i}{k}}\prod_{\pm}\Xi^{\text{norm}}_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1\pm\frac{1}{2},Z_3\mp\frac{1}{2})}(-\kappa)\\&\quad+\Big(2\sin\frac{\pi}{k}\Big)\Xi^{\text{norm}}_{k,(-1,1,k-1,Z_1,Z_3)}(-i\kappa)\Xi^{\text{norm}}_{k,(0,0,k,Z_1,Z_3)}(i\kappa)=0.\end{aligned} \tag{80}$$

Let us look at this equation at order κ^1. The exact expressions for $\frac{Z_{k,\boldsymbol{M}}(1)}{Z_{k,\boldsymbol{M}}(0)}$ evaluated at these rank variables (M_0,M_1,M_3) are cumbersome. Here for simplicity let us further set k to $k=2$, where we have [35]

$$\begin{aligned}&\frac{Z_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1,Z_3)}(1)}{Z_{k,(-\frac{1}{2},\frac{1}{2},k-\frac{1}{2},Z_1,Z_3)}(0)}=\frac{-\pi^{-1}+2iZ_1Z_3-Z_1\tan\pi Z_1-Z_3\tan\pi Z_3-i\tan\pi Z_1\tan\pi Z_3}{16\cos\pi Z_1\cos\pi Z_3}\\&\quad-\frac{2(\cos\pi Z_1+\cos\pi Z_3)-2e^{-\frac{\pi i}{4}}e^{\pi iZ_1Z_3}(e^{-\frac{\pi i(Z_1+Z_3)}{2}}+ie^{\frac{\pi i(Z_1-Z_3)}{2}}+ie^{-\frac{\pi i(Z_1-Z_3)}{2}}+e^{\frac{\pi i(Z_1+Z_3)}{2}})}{64\cos^2\pi Z_1\cos^2\pi Z_3},\\&\frac{Z_{k,(-1,1,k-1,Z_1,Z_3)}(1)}{Z_{k,(-1,1,k-1,Z_1,Z_3)}(0)}=\frac{-Z_1Z_3+i(Z_1\cot\pi Z_1+Z_3\cot\pi Z_3)-i\pi^{-1}+\cot\pi Z_1\cot\pi Z_3}{8\sin\pi Z_1\sin\pi Z_3}\\&\quad+\frac{e^{\pi iZ_1Z_3}(1-\cos\pi Z_1-\cos\pi Z_3-\cos\pi Z_1\cos\pi Z_3)}{16\sin^2\pi Z_1\sin^2\pi Z_3},\\&\frac{Z_{k,(0,0,k,Z_1,Z_3)}(1)}{Z_{k,(0,0,k,Z_1,Z_3)}(0)}=\frac{Z_1Z_3}{8\sin\pi Z_1\sin\pi Z_3}.\end{aligned} \tag{81}$$

Substituting these into the left-hand side of the bilinear equation (80) one can see that the equation is indeed satisfied at order κ.

5 Conclusion

In this paper we have reviewed the discrete symmetry of the normalized partition function $\frac{Z_{k,\boldsymbol{M}}(N)}{Z_{k,\boldsymbol{M}}(0)}$ of the $\mathrm{U}(N-M_0-M_3+k)_{k,iZ_1}\times\mathrm{U}(N-M_1-M_3+k)_{0,-iZ_1}\times\mathrm{U}(N+M_0-M_3+k)_{-k,-iZ_3}\times\mathrm{U}(N)_{0,iZ_3}$ circular quiver super Chern-Simons theory which describes multiple M2-branes probing $(\mathbb{C}^2/\mathbb{Z}_2\times\mathbb{C}^2/\mathbb{Z}_2)/\mathbb{Z}_k$. This theory enjoys discrete symmetry induced by the Hanany-Witten effect under the exchanges of the 5-branes in the type IIB brane construction of the theory. In the Fermi gas formalism of the partition function, this symmetry is realized as the discrete symmetry of the density matrix $\hat{\rho}_{\boldsymbol{M}}$ up to the similarity transformation, which corresponds to the coordinate transformation of the curve $\rho_{\boldsymbol{M}}^{-1}(x,p)=const.$. From the Fermi gas formalism, we also find that the normalized partition function has a larger symmetry which is the Weyl group of $D_5=\mathrm{SO}(10)$. Furthermore, the Fermi gas formalism with the genus-one quantum curve with $W(D_5)$ symmetry, together with the TS/ST correspondence and the $\mathfrak{q}$-uplift of the Painlevé/gauge correspondence, suggests that the grand partition function $\Xi_{k,\boldsymbol{M}}(\kappa)$ (62) satisfies the Hirota bilinear form of the $\mathfrak{q}$-discrete Painlevé equation associated with D_5, that is, $\mathfrak{q}\mathrm{P}_{\mathrm{VI}}$. We have identified the explicit expression of the bilinear equations for $\Xi_{k,\boldsymbol{M}}(\kappa)$ and provided a non-trivial check for these equations by using the exact values of $Z_{k,\boldsymbol{M}}(N)$. Our result is a non-trivial extension of the known connection between the grand partition function of $\mathrm{U}(N)_k \times \mathrm{U}(N+M)_{-k}$ ABJ theory and $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$ [22]. Interestingly, the bilinear equations we have identified are satisfied by the unnormalized grand partition function $\Xi_{k,\boldsymbol{M}}(\kappa)$ which is different from the Fredholm determinant of the curve $\mathrm{Det}(1+\kappa\hat{\rho}_{\boldsymbol{M}})$ by an overall factor $Z_{k,\boldsymbol{M}}(0)$. Actually, it was crucial to notice this fact, namely that the bilinear equations are satisfied by $Z_{k,\boldsymbol{M}}(0)$ themselves, in identifying the concrete coefficients in the bilinear equations. Since $Z_{k,\boldsymbol{M}}(0)$ is not related to the quantum curve, this fact may also suggest that there is another way to understand the relation between the matrix models and $\mathfrak{q}$-Painlevé equations which do not rely on the Fermi gas formalism and the TS/ST correspondence.

At the moment, although there are many pieces of evidence for the relations between the matrix models and $\mathfrak{q}$-Painlvé equations, a direct proof is still missing. However, there are several analogous relations between matrix models and integrable systems which are actually proved. For example, the relation between the ABJ matrix model and $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$ was proved in the "dual 4d limit" [49] where the ABJ matrix model reduces to the O(2) matrix model [50] and the $\mathfrak{q}\mathrm{P}_{\mathrm{III}_3}$ reduces to the Painlevé III_3 differential equation [51]. There is also a related study for higher Painlevé equation [52]. It would be interesting to prove the $\mathfrak{q}$-Painlevé bilinear relations for the grand partition functions directly from the matrix models hinted by these known results. Finding such a proof would also give an indirect proof for the TS/ST correspondence, as claimed in [49] for $\mathfrak{q} \to 1$ limit.

Data Availability

No data are available.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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