Journal of Holography Applications in Physics
Volume 3, Issue 1, Winter 2023, 49-56
©Available online at http://jhap.du.ac.ir
DOI:10.22128/jhap.2023.654.1041
Online ISSN: 2783-3518

Journal of Holography Applications in Physics

Regular article

Geodesic motion in the spacetime of a $(2+1)$ D black hole conformally coupled to a massless scalar

Saheb Soroushfar
Faculty of Technology and Mining, Yasouj University, Choram 75761-59836, Iran;
Email: soroush@yu.ac.ir

Received: January 25, 2023; Revised: February 25, 2023; Accepted: March 10, 2023.

Abstract. In this paper, we consider a $(2+1)$ D black hole conformally coupled to a massless scalar. Then the geodesic motion of test particles and light rays in the vicinity of the spacetime of this black hole is studied. Moreover, the geodesic equations are solved analytically according to Weierstrass elliptic and derivatives of Kleinian sigma hyperelliptic functions. Also, the possible orbits are discussed and classified according to the particle's energy and angular momentum.

Keywords: Black hole; Geodesic motion; Elliptic function; Analytical solution.

1 Introduction

The beginning of the exact solutions of three-dimensional black holes can be considered with the famous work of Banados, Teitelboim, and Zanelli (BTZ) black hole [1, 2]. After that, different types of solutions were obtained by considering different scalar fields (coupled scalar, regular scalar, asymptotic coupled scalar and etc) [3, 4, 5, 6, 7]. Also, several three-dimensional black hole solutions for different cases (static, charged, dilaton gravity, $f(R)$ gravity, and so on) have been obtained [8, 9, 10, 11, 12]. Moreover, a better understanding of the physics of black holes by studying them in lower dimensions can be useful. One of the most experimental ways to investigate the gravitational field of black holes is to study the movement of particles and light around them. It is true that the effects of gravity can be investigated by means of numerical solutions, but a comprehensive study should be done by means of analytical solutions. Analytical solutions help a lot to study experimental predictions, including (light deviation, gravitational time delay, perihelion shift, and Lens-Teering effect). The geodesic equations of many different space-times, whether in the standard theory of general relativity or in alternative theories, have been studied by various researchers and tried to be solved analytically [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. In this paper, elliptic and hyperelliptic functions are used to solve the geodesic equations analytically. In section 2, a brief review of a $(2 + 1)$ D black hole conformally coupled to a massless scalar is provided. In section 3, the analytical solution of the equations of motion for both timelike and null geodesic equations is studied. Then, using analytical solutions and the effective potential, some of the possible orbits are demonstrated. Moreover, the conclusion is represented in section 4.

2 $(2 + 1)$ D black hole conformally coupled to a massless scalar

In this section, we briefly study the properties of the metric of (2+1) D black hole conformally coupled to a massless scalar. The action used in this theory can be written as follows [24, 25]

$$S = \int d^3x\sqrt{-g}\left[R + \frac{2}{l^2} - \nabla_\mu\varphi\nabla^\mu\varphi - \frac{1}{8}R\varphi^2\right], \tag{1}$$

where l is length scale related to cosmological constant ($l^2 = -\frac{1}{\Lambda}$), R is the Ricci scalar, and ϕ is the conformal scalar field. the field equations corresponding to metric and also the scalar field, are respectively as follows [24, 25]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}l^{-2} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{8}(g_{\mu\nu}\nabla^\mu\nabla_\mu - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2, \tag{2}$$

$$(\nabla^\mu\nabla_\mu - \frac{1}{8}R)\phi = 0. \tag{3}$$

From the above equations, it can be concluded that

$$R = -\frac{6}{l^2}. \tag{4}$$

So, static and circularly symmetric three dimensional metric takes the following form

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + r^2d\phi^2, \tag{5}$$

where $g(r)$ is a general metric function, which can be expressed as follows

$$g(r) = \frac{r^2}{l^2} - \frac{\xi_1}{r} - \xi_2, \tag{6}$$

where ξ_1 and ξ_2 are the constants of integration and can be written as [24]

$$\xi_1 = \frac{2b^3}{l^2}, \qquad \xi_2 = \frac{3b^2}{l^2}. \tag{7}$$

Therefor,

$$g(r) = \frac{(r+b)^2(r-2b)}{l^2r}, \tag{8}$$

where b is a constant. The radius of the horizon (r_h) satisfies the condition $f(r)|_{r=r_h} = 0$ and so, $b = \frac{r_h}{2}$. The details of this solution are shown in [24, 25].

3 Analytical solution of geodesic equations

In this section, the geodesic equations and analytical solutions are studied. The geodesic motion is described by the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = 0, \tag{9}$$

where $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol. The conserved energy and angular momentum as a constant of motions are obtained by the normalization condition $\frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{1}{2}\epsilon$, where for massive particles $\epsilon = 1$ and for light $\epsilon = 0$,

$$E = g_{tt}\frac{dt}{ds}, \tag{10}$$

$$L = g_{\varphi\varphi}\frac{d\varphi}{ds} = r^2\frac{d\varphi}{ds}. \tag{11}$$

So, the geodesic equations can be obtained as

$$(\frac{dr}{ds})^2 = E^2 - (\frac{(r+b)^2(r-2b)}{l^2 r})(\epsilon + \frac{L^2}{r^2}), \tag{12}$$

$$(\frac{dr}{d\varphi})^2 = \frac{r^4}{L^2}(E^2 - \frac{(r+b)^2(r-2b)}{l^2 r})(\epsilon + \frac{L^2}{r^2})). \tag{13}$$

Equations (12) and (13) give a description of the dynamics of the geodesic motion. The effective potential can be obtained from Eq.(12) as

$$V_{eff} = (\frac{(r+b)^2(r-2b)}{l^2 r})(\epsilon + \frac{L^2}{r^2}). \tag{14}$$

Plots of effective potential are shown in Fig. (1). In the following, it is convenient to use dimensionless quantities as

$$\tilde{r} = \frac{r}{m}, \qquad \tilde{L} = \frac{L}{m}, \qquad \tilde{b} = \frac{b}{m}, \qquad \tilde{l} = \frac{l}{m}, \tag{15}$$

and rewrite Eq.(13) as

$$(\frac{dr}{d\phi})^2 = -\frac{\tilde{L}\tilde{r}^6}{\tilde{l}^2} + \left(E^2\tilde{L} + 3\frac{\tilde{L}\tilde{b}^2}{\tilde{l}^2} - \tilde{l}^{-2}\right)\tilde{r}^4 + 2\frac{\tilde{L}\tilde{b}^3\tilde{r}^3}{\tilde{l}^2} + 3\frac{\tilde{b}^2\tilde{r}^2}{\tilde{l}^2} + 2\frac{\tilde{b}^3\tilde{r}}{\tilde{l}^2} = R(\tilde{r}). \tag{16}$$

The necessary condition for the existence of a geodesic is $R(\tilde{r}) \geq 0$. Thus, the zeros of $R(\tilde{r})$, determine the type of geodesic.

3.1 Null geodesics

For $\epsilon = 0$ and $u = \frac{1}{r}$, Eq.(16) is of elliptic type $P_3(u) = \sum_{i=0}^{3} a_i u^i$. therefore, With the substitution $u = \frac{1}{a_3}(4y - \frac{a_2}{3})$, can be transformed to the Weierstrass form as

$$(\frac{dy}{d\varphi})^2 = 4y^3 - g_2 y - g_3 = P_3(y), \tag{17}$$

in which, $g_2 = \frac{a_2^2}{12} - \frac{a_1 a_3}{4}$ and $g_3 = \frac{a_1 a_2 a_3}{48} - \frac{a_0 a_3^2}{16} - \frac{a_2^3}{216}$ are the Weierstrass invariants. So, the analytical solution of Eq.(17) is given by

$$y(\varphi) = \wp(\varphi - \varphi_{in}), \tag{18}$$

and therefore,

$$\tilde{r}(\varphi) = \frac{a_3}{4\wp(\varphi - \varphi_{in}; g_2, g_3) - \frac{a_2}{3}}, \tag{19}$$

where

$$\varphi_{in} = \varphi_0 + \int_{y_0}^{\infty} \frac{dz}{\sqrt{4y^3 - g_2 - g_3}}, y_0 = \frac{a_3}{4\tilde{r_0}} + \frac{a_2}{12}. \tag{20}$$

depends only on the initial values φ_0 and r_0.

3.2 Timelike geodesics

For $\epsilon = 1$ and $u = \frac{1}{r}$, Eq.(16) is of hyper-elliptic type as

$$\begin{aligned} (u\frac{du}{d\varphi})^2 &= (\beta(2-\gamma\beta))u^5 + (3\beta\gamma - 1)u^4 \\ &+ (\beta(2-\beta\gamma)\mathcal{L} - \gamma)u^3 + (E^2\mathcal{L} + 3\beta\gamma\epsilon\mathcal{L} - \epsilon\mathcal{L} + k)u^2 \\ &- \gamma\epsilon\mathcal{L}u + k\epsilon\mathcal{L} = p_5(u), \end{aligned} \tag{21}$$

and the solution of this equation can be written as

$$u(\varphi) = -\frac{\sigma_1}{\sigma_2}\varphi(\sigma), \tag{22}$$

in which σ_i is the i-th derivative of the Kleinian sigma function in two variables

$$\sigma(z) = Ce^{z^t kz}\theta[K_\infty](2\omega^{-1}z; \tau), \tag{23}$$

which is given by the vector of Riemann constants with base point at infinity $2[g, h] = (0, 1)^t + (1, 1)^t\tau$ and the Riemann θ-function with characteristic $[g, h]$. So, the analytical solution of Eq.(16) is

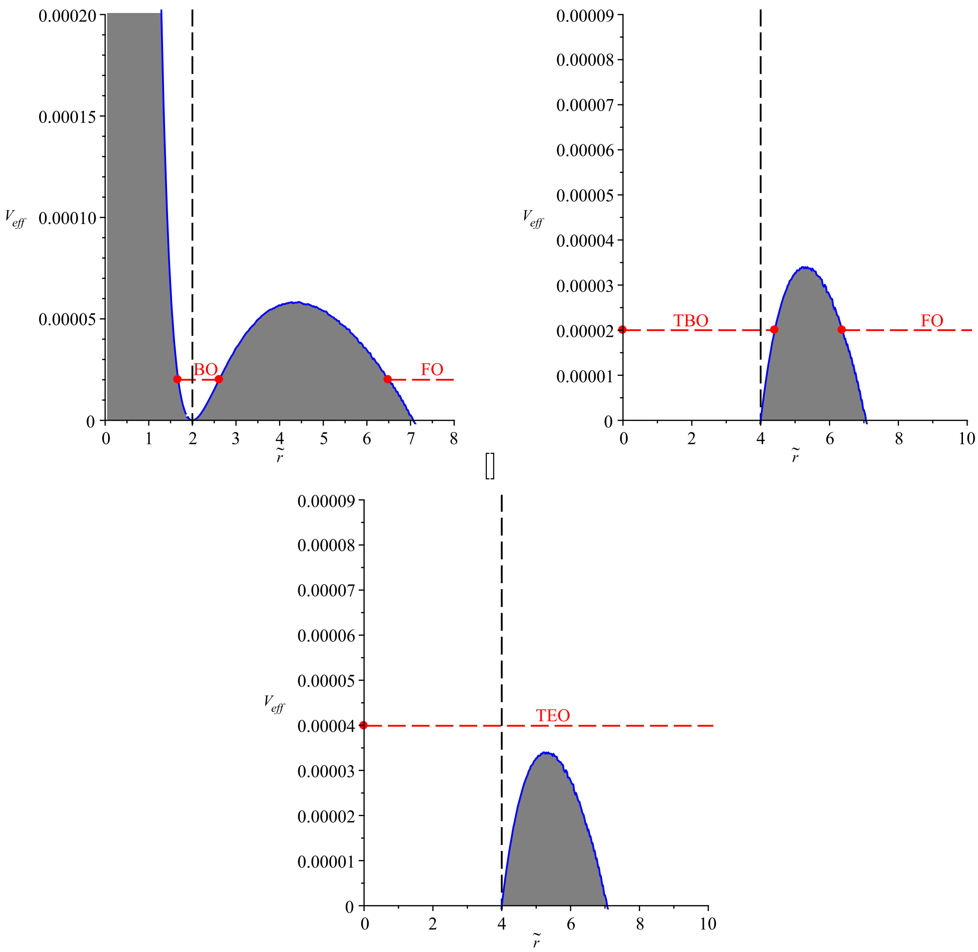
$$\tilde{r}(\varphi) = -\frac{\sigma_2}{\sigma_1}\varphi_\sigma. \tag{24}$$

3.3 Orbits

In this section, using obtained analytical solution and the figures of the effective potential Fig. 1, some of the possible orbits are shown in Fig. 2. The number of real and positive zeros of $R(\tilde{r})$ characterize the possible orbits. Therefore, here different types of orbits can be identified as

1. Flyby orbits: r starts from ∞, then approaches a periapsis $r = r_p$ and goes back to ∞.
2. Bound orbits: r oscillates between two boundary values $r_p \leq r \leq r_a$ with $0 < r_p < r_a < \infty$.
3. Terminating bound orbits: r starts in $(0, r_a]$ for $0 < r_a < \infty$ and falls into the singularity at $r = 0$.
4. Terminating escape orbits: r comes from ∞ and falls into the singularity at $r = 0$.

Examples of such orbits are demonstrated in Fig. 2.



[] [] []

Figure 1: Variations of the effective potential in terms of horizon radius. The horizontal red dash line represents the energy. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

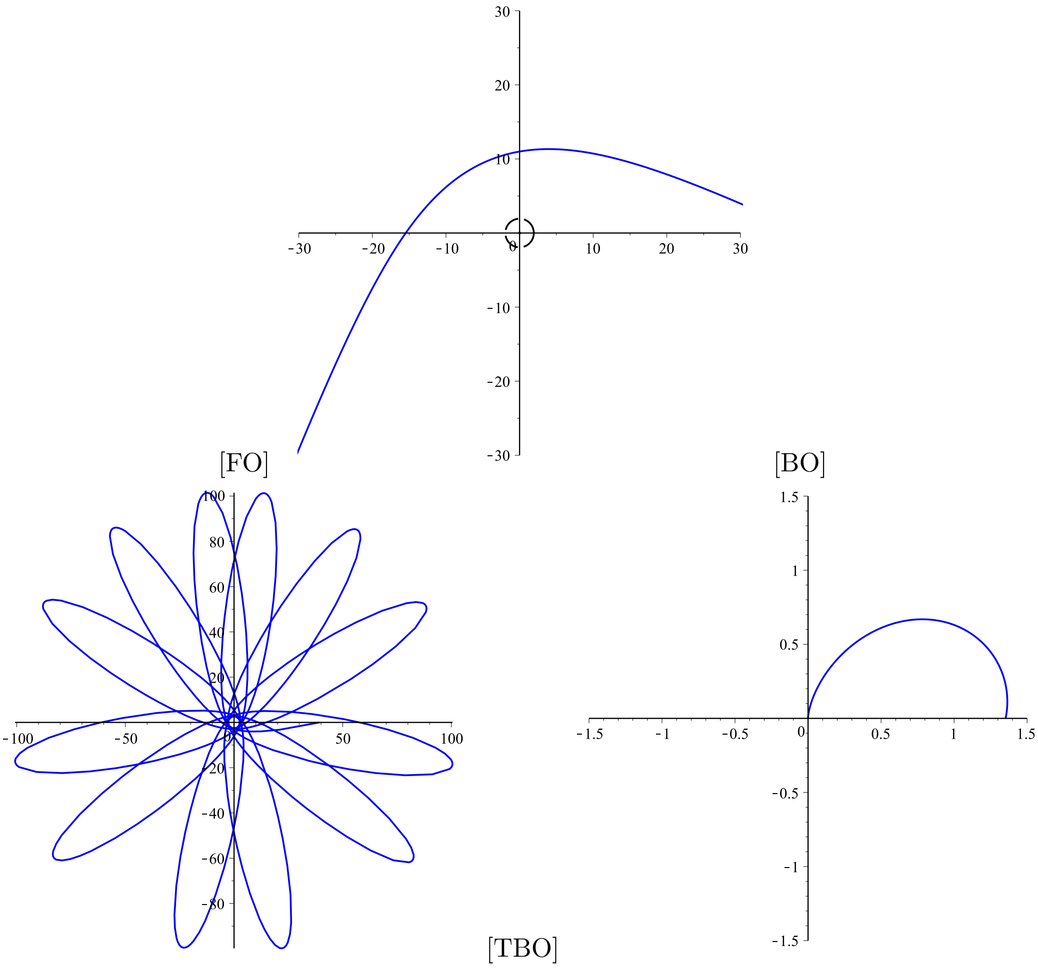


Figure 2: Examples of different types of orbits.

4 Conclusion

In this paper, the motion of massive and massless test particles in the vicinity of a $(2+1)$ D black hole conformally coupled to a massless scalar was investigated. Then, the equations of geodesic motions were obtained and solved according to Weierstrass elliptic and Kleinian sigma hyper-elliptic functions. Moreover, with the help of these analytical solutions and the effective potential, the set of orbit types was classified. For timelike and null geodesics, various orbits such as FO, BO, TBO, and TEO are possible. These results can be useful information for orbits around heavy objects, including light deflection, periastron shift, and else.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

Funding

This research did not receive any grant from funding agencies in the public, commercial, or non-profit sectors.

References

[1] M. Banados, C. Teitelboim and J. Zanelli, The Black hole in three-dimensional spacetime, Phys. Rev. Lett. **69**, 1849-1851 (1992) [arXiv:hep-th/9204099 [hep-th]].

[2] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the (2+1) black hole, Phys. Rev. D **48**, 1506-1525 (1993) [erratum: Phys. Rev. D 88, 069902 (2013)].

[3] M. Cataldo, N. Cruz, S. del Campo and A. Garcia, (2+1)-dimensional black hole with Coulomb - like field, Phys. Lett. B **484**, 154 (2000) [arXiv:hep-th/0008138 [hep-th]].

[4] C. Martinez and J. Zanelli, Conformally dressed black hole in (2+1)-dimensions, Phys. Rev. D **54**, 3830 (1996) [grqc/9604021].

[5] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, Black holes and asymptotics of 2+1 gravity coupled to a scalar field, Phys. Rev. D **65**, 104007 (2002) [hep-th/0201170].

[6] F. Correa, C. Martinez and R. Troncoso,Scalar solitons and the microscopic entropy of hairy black holes in three dimensions, JHEP **1101**, 034 (2011). [arXiv:1010.1259 [hep-th]].

[7] F. Correa, C. Martinez and R. Troncoso, Hairy Black Hole Entropy and the Role of Solitons in Three Dimensions, JHEP **1202**, 136 (2012). [arXiv:1112.6198 [hep-th]].

[8] W. Xu and D. C. Zou, (2+1) Dimensional charged black holes with scalar hair in Einstein Power Maxwell Theory, Gen. Rel. Grav. **49**, no.6, 73 (2017).

[9] W. Xu and L. Zhao, Charged black hole with a scalar hair in (2+1) dimensions, Phys. Rev. D **87**, no.12, 124008 (2013) [arXiv:1305.5446 [gr-qc]].

[10] M. Cataldo and A. Garcia, Regular (2+1)-dimensional black holes within nonlinear electrodynamics, Phys. Rev. D **61**, 084003 (2000) [arXiv:hep-th/0004177 [hep-th]].

[11] S. Habib Mazharimousavi, M. Halilsoy and T. Tahamtan, Regular charged black hole construction in 2+1 -dimensions, Phys. Lett. A **376**, 893-898 (2012).

[12] Y. He and M. S. Ma, (2 + 1)-dimensional regular black holes with nonlinear electrodynamics sources, Phys. Lett. B **774**, 229-234 (2017) [arXiv:1709.09473 [gr-qc]].

[13] S Chandrasekhar (Oxford: Clarendon press) (1985).

[14] E Hackmann V Kagramanova J Kunz and C Lammerzahl, Physical Review D **78** 124018 (2008) Erratum [Physical Review D **79** 029901 (2009)].

[15] E Hackmann and C Lammerzahl, Physical Review Letters **100** 171101 (2008).

[16] E Hackmann C Lammerzahl V Kagramanova and J Kunz Physical Review D **81** 044020 (2010).

[17] S Soroushfar R Saffari J Kunz and C Lammerzahl Physical Review D **92** 044010 (2015).

[18] B Hoseini R Saffari S Soroushfar J Kunz and S Grunau Physical Review D **94** 044021 (2016).

[19] V Z Enolski E Hackmann V Kagramanova J Kunz and C Lammerzahl Journal of Geometry and Physics **61** 899 (2011).

[20] V Kagramanova and S Reimers Physical Review D **86** 084029 (2012). V Diemer J Kunz C Lammerzahl and S Reimers Physical Review D 89124026 (2014).

[21] S Soroushfar R Saffari S Kazempour S Grunau and J Kunz Physical Review D **94** 024052 (2016).

[22] S. Soroushfar, R. Saffari, and A. Jafari, Phys. Rev. D **93**, 104037 (2016).

[23] S. Soroushfar, M. Afrooz, Analytical solutions of the geodesic equation in the spacetime of a black hole surrounded by perfect fluid in Rastall theory. Indian J Phys 96, 593-607 (2022).

[24] C. Martinez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996).

[25] H.K. Sudhanshu, S. Upadhyay, D.V. Singh, et al. Corrected Thermodynamics of (2+1)D Black Hole Conformally Coupled to a Massless Scalar. Int J Theor Phys **61**, 248 (2022).