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Holographic dual picture of a modified Horndeski black hole

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Abstract. The usual Horndeski black hole does not have $P-V$ critical points, hence does not show any phase transitions. In this article, a particular modified Horndenski black hole is considered to study the $P-V$ diagram and the phase transitions. We show that this modified Horndeski black hole solution satisfies the *Ist* order phase transition. We also show that the modified Horndeski black hole is a holographic dual of a *Van der Waals*(VdW) fluid. Finally, we study the thermodynamics of a modified Horndeski black hole based on the equation of state originating from the slope of temperature versus entropy. This new prescription provides us a simple and powerful way to study the critical behavior and the phase transition of black holes. The analytical interpretation of possible phase transition points leads us to set some nonphysical range on the horizon radius for the black hole.

Keywords: Thermodynamics; Holography; Horndeski black hole.

1 Introduction

Black holes are a remarkable prediction of general theory of relativity (GR). However, the quantum aspects of black holes can be studied only in the framework of quantum gravity. Almost all approached to quantum gravity introduce the minimal length scale, believed to be of the order of Planck Length ℓ_P in the background geometry of space-time [1]. Effects of the minimal length scale are expected to be universal [2], and have a consequence on all quantum system even at low energies (see e.g. [3, 4]). Any quantum theory of gravity, should be consistent with black hole physics. In order to describe the quantum aspects of classical black holes, we need to apply some modifications to the ordinary black hole solutions. Here, we consider the Horndeski black hole metric specifically in Anti-de Sitter space (AdS) [5, 6, 9, 10, 7, 8]. For such black holes, it has been explicitly shown that the Hawking-Page transition exists only for some specific values of the Horndeski parameters, thus making small black holes more stable than larger ones [11, 12]. It was Hawking and Bekenstein, who first established the working relationship between black hole mechanics which typically include the quantities like surface gravity, mass, and area of a black hole to its thermodynamic quantities like temperature, energy, and entropy [13, 14, 15, 16, 17, 18, 19, 20]. In fact, some attempts have been made to understand the holographic entanglement entropy as a function of Horndeski parameters with an infinite strip region of the boundary [6]. The idea of including the variation of cosmological constant Λ in the first law of black hole thermodynamics has been attended recently by several authors and applied to several black holes.

We can start by identifying the pressure in natural units with the following expression

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi\ell^2} = \frac{3}{8\pi\ell^2}, \tag{1}$$

where ℓ is the length of AdS space and d stands for the number of spacetime dimensions; $d = 4$ corresponds to the four spacetime dimensions. The conjugate variable of pressure is defined to be the thermodynamic volume of the black hole [44] given by

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,J}, \tag{2}$$

where M is the black hole mass. For a Schwarzschild black hole of radius r_h with area $4\pi r_h^2$, the thermodynamic volume coincidentally turns out to be $4/3\ \pi r_h^3$. For black holes in general the thermodynamic volume is not geometrically related to their area [16].

It may also be noted that the presence of a negative cosmological constant in Eq. (1) will have an interesting implications on black hole thermodynamics. The negative cosmological constant is a specific characteristic of the AdS space thus having a huge advantage in holography and AdS/CFT correspondence. The critical point in charged AdS black holes shows that such black hole has a VdW fluid behavior [21]. It has been found that the VdW fluid is the holographic dual of RN AdS black hole [22]. So, by using the holographic principles, one can study AdS black holes via a Van der Waals fluid and understand $P-V$ criticality [21]. Also, it is found that spinning Kerr-AdS black holes in five dimensions, behave as VdW fluid [23].

The VdW phase transition and $P-V$ criticality of AdS black holes in the general framework are already discussed in Refs. [24, 25], which is extended to the massive gravity in Ref. [26], and found that presence of logarithmic correction [27] is necessary to have a holographic dual of VdW fluid [28, 29].

The equation of state of VdW fluid is a popular closed form modification of the ideal gas law. It approximates the behavior of real fluids by taking into account the nonzero size of molecules and the attraction between them. It is often used to describe the qualitative features of the Liquid-gas phase transition. In that case, the equation reads,

$$\left(P+\frac{a}{v^2}\right)(v-b)=kT, \tag{3}$$

where $v=\frac{V}{N}$ is the specific volume of the fluid and k is the Boltzmann constant. The constant $b>0$ takes into account the nonzero size of the molecules of a given fluid, whereas the constant $a>0$ ensures the attraction between them. One can expand this equation to write it as a cubic equation for v,

$$Pv^3-(kT+bP)\,v^2+av-ab=0. \tag{4}$$

In order to investigate the $P-V$ critical points of gas, we need to apply the following conditions,

$$\left(\frac{\partial P}{\partial v}\right)_{S,Q,J}=0, \quad \left(\frac{\partial^2 P}{\partial v^2}\right)_{S,Q,J}=0. \tag{5}$$

In the presence of a negative cosmological constant, the asymptotically AdS black hole admits a gauge duality description with dual thermal field theory. Such theory leads us to an interesting phenomenon which is called Hawking and Page phase transition [22, 30, 31, 32, 33]. This article is devoted to the phase transitions taking place near the critical point. There are different approaches to investigating the phase transition, some of which have been used to study the behavior of the heat capacity in different ensembles. Here, we use two major approaches to examine the phase transition. In the first approach, the changes of the signature in the heat capacity represent the phase transitions, and hence the roots of heat capacity have decisive roles. In the second approach, the divergences of the heat capacity indicate the phase transitions, and hence the singular points of the heat capacity become more important [34, 35]. The heat capacity is an interesting thermodynamic quantity to determine the stability and instability of the black hole. In general, black hole heat capacity is always negative which shows that the black hole is unstable and has Hawking radiation. But with the presence of charge and rotation parameters of the black hole, the heat capacity can change the sign, and become positive, thus the phase transition occurs. In this article, we use a novel method to study the phase transitions in which the critical behavior of the VdW gas is obtained by using the slope of T versus S [36, 37, 38].
According to the standard methods, in the usual extended phase transition space, one should calculate firstly $T=\frac{\partial M}{\partial S}$ to obtain the equation of state. The other calculations then take place by using the state equation. Instead, here applying the new method, we use $\frac{\partial T}{\partial S}=\frac{\partial^2 M}{\partial S^2}$ to find the equation of state. Following that, the thermodynamic quantities of our physical system can be studied.

2 Modified Horndeski black hole solution

We begin with the following action [7],

$$S=\int d^4x\sqrt{-g}\left[\left(\zeta+\beta\sqrt{\frac{(\partial\phi)^2}{2}}\right)R-\frac{\eta}{2}(\partial\phi)^2-\frac{\beta}{\sqrt{2(\partial\phi)^2}}\left[(\triangle\phi)^2-(\nabla_\mu\nabla_\nu\phi)^2\right]\right], \tag{6}$$

where η and β are dimensionless parameters, they can be absorbed into the scalar field by means of a redefinition. The coefficient ζ gives the Einstein- Hilbert part of the action, which is $\zeta = M_{pl}^2/16\pi$. The field equations from the equation (6) admit a static, spherically symmetric and asymptotically flat solution [37], given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{7}$$

where,

$$f(r) = 1 - \frac{2M}{r} - \frac{\beta^2}{2\zeta\eta r^2}. \tag{8}$$

The parameters β and η should share the same sign, i,e, $\beta > 0$ and $\eta > 0$.
Here, the equation (8) lead us to obtain the following temperature,

$$T = \frac{\kappa_+}{2\pi} = \frac{1}{4\pi}\frac{df(r)}{dr}|_{r=r_+} = \frac{1}{4\pi r_+}\left(1 - \frac{\gamma}{r_+^2}\right) \tag{9}$$

where $\gamma = \frac{\beta^2}{2\zeta\eta}$.
Since the equation (9) does not satisfy the conditions given by Eqs. (3) and (5), it can be concluded that there is no $P-V$ critical behavior. The correspondence of fluid dynamics and the black hole equation of state provides us an opportunity to connect the two theories from their VdW behavior. When we stay on the fluid dynamic side, we have VdW behavior. But, from the black hole side, there is no a such equation of state. So, in that case, we modify the black hole solution (8) such that the dynamical properties of the black hole coincide exactly with the corresponding fluid, without the need for the additional matter to be added to the action. In order to observe $P-V$ critical behavior, we change the Horndeski black hole with the following ansatz [39],

$$f(r) = 1 - \frac{2M}{r} - \frac{\gamma}{r^2} + h(r, P), \tag{10}$$

where the function $h(r, P)$ is needed to be fixed such as to guarantee a black hole solution with suitable thermal properties as well as VdW behavior. As already noted we do not add additional matter fields, thus the action remains unchanged. Also the metric with the above modification is a solution of the Einstein field equations with a given energy momentum source, $G_{ab} + \Lambda g_{ab} = 8T_{ab}$. We note here the corresponding energy-momentum source from the modified metric should be satisfied by weak, strong, and dominant conditions [40]. These conditions are known as energy conditions which are satisfied by our modified ansatz of metric background.
Now we try to obtain the modified metric of this black hole. By using the Euclidean trick and equation (10) in (9), one can identify the black hole temperature as [39, 41],

$$T = \frac{1}{4\pi}\left(\frac{1}{r_+} + \frac{\gamma}{r_+^3} + \frac{h(r_+, P)}{r_+} + h'(r_+, P)\right). \tag{11}$$

By using the following VdW equation of state,

$$T = \left(P - \frac{a}{\upsilon^2}\right)(\upsilon + b) = P\upsilon - Pb + \frac{a}{\upsilon} - \frac{ab}{\upsilon^2}, \tag{12}$$

one can obtain T as,

$$T = 2Pr_+ - Pb + \frac{a}{2r_+} - \frac{ab}{4r_+^2}. \tag{13}$$

where $\upsilon = 2r_+$can be identified as specific volume. Comparing (11) and (13), we can rewrite the following expression,

$$\frac{1}{4\pi}\left(\frac{1}{r_+}+\frac{\gamma}{r_+^3}+\frac{h(r_+,P)}{r_+}+h'(r_+,P)\right)-2Pr_++Pb-\frac{a}{2r_+}+\frac{ab}{4r_+^2}=0. \tag{14}$$

In order to obtain the $h(r_+,P)$, we can rearrange $h(r_+,P)$ as following [42, 43],

$$\begin{aligned} h(r_+,P) &= A(r_+)-PB(r_+) \\ h'(r_+,P) &= A'(r_+)-PB'(r_+). \end{aligned} \tag{15}$$

From (14) and (15), one can obtain the following expression,

$$P\left(b-2r_+-\frac{B(r_+)}{4\pi r_+}-\frac{B'(r_+)}{4\pi}\right)-\left(\frac{a}{2r_+}-\frac{ab}{4r_+^2}-\frac{1}{4\pi r_+}-\frac{\gamma}{4\pi r_+^3}-\frac{A(r_+)}{4\pi r_+}-\frac{A'(r_+)}{4\pi}\right)=0. \tag{16}$$

Here, two terms must be independently zero. So, we have,

$$b-2r_+-\frac{B(r_+)}{4\pi r_+}-\frac{B'(r_+)}{4\pi}=0, \tag{17}$$

one can obtain $B(r_+)$ as,

$$B(r_+)=4\pi\left(b\frac{r_+}{2}-\frac{2}{3}r_+^2\right). \tag{18}$$

Again, the second term will be,

$$A'(r_+)+\frac{A(r_+)}{r_+}=\frac{(2\pi a-1)}{r_+}-\frac{\pi ab}{r_+^2}-\frac{\gamma}{r_+^3}, \tag{19}$$

and $A(r_+)$ is given by the following equation,

$$A(r_+)=(2\pi a-1)-\pi ab\frac{\ln(r_+)}{r_+}+\frac{\gamma}{r_+^2}. \tag{20}$$

So, from the previous ansatz, we obtain $h(r_+,p)$ as follows,

$$h(r_+,p)=(2\pi a-1)-\pi ab\frac{\ln(r_+)}{r_+}+\frac{\gamma}{r_+^2}+\frac{2}{3}\pi P\left(4r_+^2-3br_+\right). \tag{21}$$

So, the modified metric function is given by,

$$f(r_+)=2\pi a-\frac{2M}{r_+}-\pi ab\frac{\ln(r_+)}{r_+}+\frac{2}{3}\pi P\left(4r_+^2-3br_+\right). \tag{22}$$

Hence, we modified the Horndeski black hole by a new definition of $h(r_+,P)$ ansatz. We plot in Fig. 1 (a) the function $f(r_+)$ in terms of horizon radius for the corresponding black hole. Here, it can be seen that there exists a critical point for $f(r_+)$ which decreases as long as M mass of the black hole increases in the critical point. It is clear from the black dashed line of Fig. 1 (b). By increasing the physical mass, the function $f(r_+)$ decreases and increases before and after the critical horizon radius respectively.

In the next sections, we will use equation (22) and obtain the $P-V$ critical points, in that case, we employ the ordinary and the new method.

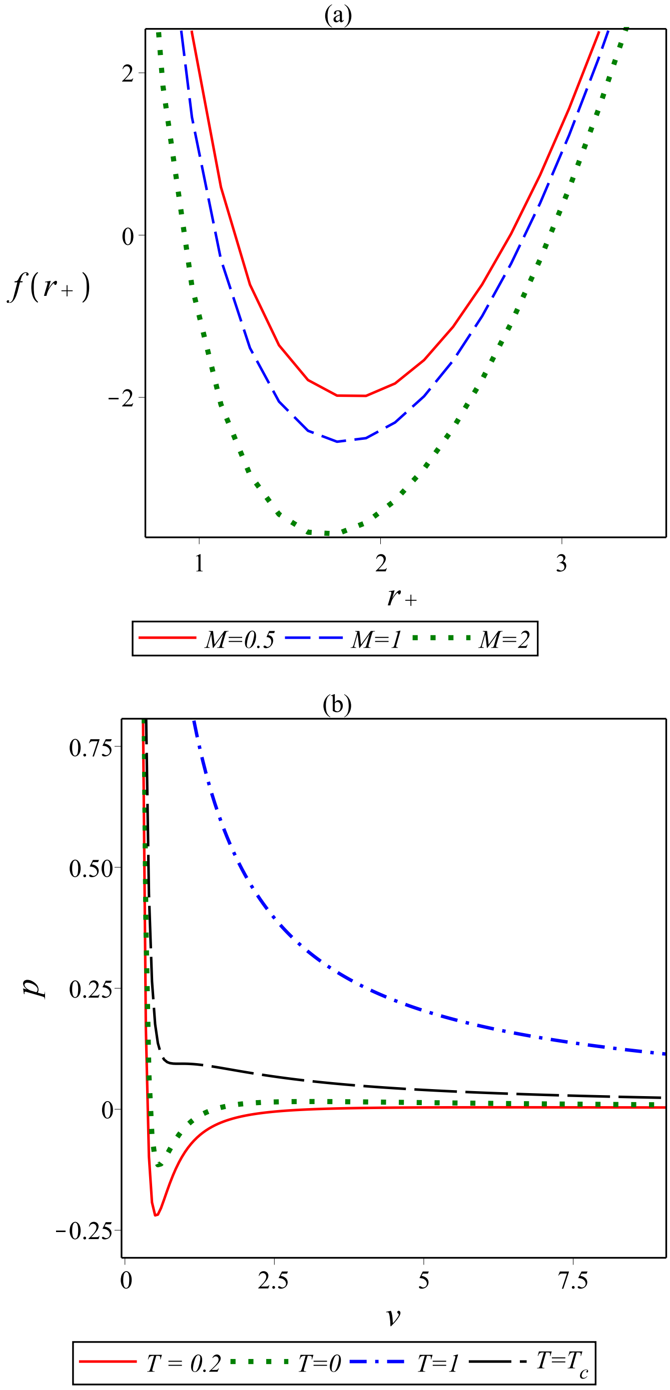

Figure 1: (a) Horizon radius with a variation of black hole mass M and $a = 1$, $b = 4$, $P = 0.2$. (b) Pressure in terms of ν for $a = 0.5$, $b = 1.5$, $a = 1.1$ and possible values of T.

3 Usual critical points

Now, we are going to investigate the $P-V$ critically for the modified metric. In that case, we need some thermodynamical quantities which play an important role in the study of the $P-V$ critical system. As we know, the position of the black hole event horizon is determined with $f(r_+) = 0$. The parameter M represent the ADM mass of the black hole, where the physical mass is given by,

$$M = \pi a r_+ - \frac{\pi ab}{2}\ln r_+ + 2\pi P(\frac{2r_+^3}{3} - b\frac{r_+^2}{2}). \tag{23}$$

We see in Fig. 1 (b) the behavior of physical mass in terms of horizon radius. Note that the values of physical mass decrease and increase before and after the minimum point respectively. Also, we see that there exists a critical point for physical mass which decreases as well as increasing the coefficients a and b.
Also, we have to calculate the Hawking temperature as follows,

$$T = \frac{a}{2r_+} - \frac{ab}{4r_+^2} + P(2r_+ - b). \tag{24}$$

The black hole entropy is given by,

$$S = \frac{\mathcal{A}}{4} = 4\pi r_+^2, \tag{25}$$

and the pressure by,

$$P = \frac{1}{\upsilon - b}\left(T - \frac{a}{\upsilon} + \frac{ab}{\upsilon^2}\right). \tag{26}$$

We can use the equation (5) to investigate the $P-V$ diagram of the modified Horndeski black hole. By using the equations (5) and (26), the following critical points are obtained

$$P_c = \frac{a}{27b^2}\ , \quad T_c = \frac{8a}{27b} \quad \text{and} \quad \upsilon_c = 3b. \tag{27}$$

The above equations give us the following expression

$$\rho = \frac{P_c \upsilon_c}{T_c} = \frac{3}{8}, \tag{28}$$

where ρ is a universal constant in an ideal gas. The above product is equal to $\frac{3}{8}$, in that case, we find an interesting relationship which is exactly the same as the Van der Walls fluid, and it is a universal number predicted for the modified Horndeski black hole.
The typical behavior of the $P-V$ diagram corresponding to the modified Horndeski black hole is plotted in Fig. 1 (b). We can see black dashed lines in Fig. 1 (b) to find that a modified Horndeski black hole is also the dual of Van der Waals fluid.
As we know there are several methods to investigate thermal stability and phase transition. We need quantities which play an important role in the study of stability system as Gibbs free energy and heat capacity. When the Gibbs free energy is negative ($G < 0$), the system has global stability. Also, when it is positive ($G > 0$), the system has local stability. In order to discuss the global and local stability of the black hole we need to calculate the Gibbs free energy which is given by,

$$G = M - ST = \pi ab - \pi a r_+ - \frac{\pi ab}{2}\ln r_+ - \pi P r_+^2\left(\frac{8}{3}r_+ - b\right), \tag{29}$$

The analysis of the Gibbs free energy for the modified Horndeski black hole shows that by increasing the horizon radius, the Gibbs free energy has global stability. For different values of parameters a and b there exists a critical point for the Gibbs free energy which increases and decreases before and after the critical point respectively.
As we know, heat capacity is an important measurable physical quantity which can determine the stability of the system. Here, we study the two types of phase transitions; the first and second order.
As we discussed in the previous section, the phase transition of the first order occurs when the temperature vanishes. So, to putting $T = 0$ in equation (26) we will have,

$$P|_{T=0} = -\frac{a}{4r_+^2}. \tag{30}$$

It means that $T = 0$ indicates a bound point between nonphysical ($T < 0$) and physical ($T > 0$) regimes. Also, the heat capacity has the following relation with mass M, entropy S, and temperature T,

$$C = T\left(\frac{\partial S}{\partial T}\right) = \left(\frac{\partial M}{\partial S}\right) / \left(\frac{\partial^2 M}{\partial S^2}\right). \tag{31}$$

By using the equations (24) and (25), we obtain the heat capacity as,

$$C = \left(2a\sqrt{\pi}S^{\frac{3}{2}} - \pi abS + 4S^2P(\frac{2}{\sqrt{\pi}}S^{\frac{1}{2}} - b)\right) / \left(-a\sqrt{\pi}S^{\frac{1}{2}} + \pi ab + \frac{P}{\sqrt{\pi}}S^{\frac{3}{2}}\right). \tag{32}$$

If $C > 0$, the black hole is in a stable state, and if $C < 0$, the black hole is in an unstable state. As regards the change of sign in specific heat with asymptotic behavior, it represents the phase transition between unstable/stable states. $C = 0$ corresponds to the phase transition of the VdW fluid similar to the critical point discussed above which leads to the following equations,

$$\begin{aligned} 2a\sqrt{\pi}S^{\frac{3}{2}} - \pi abS + 4S^2P(\frac{2}{\sqrt{\pi}}S^{\frac{1}{2}} - b) = 0, \\ 2ar_+ - ab + 4Pr_+^2(2r_+ - b) = (a + 4r_+^2P)(2r_+ - b) = 0, \end{aligned} \tag{33}$$

in agreement with Eq. (30). As we said, the phase transition of the second order is associated with divergence points of the specific heat, implying $\frac{\partial^2 M}{\partial S^2} = 0$. Therefore,

$$8Pr_+^3 - 2ar_+ + ab = 0 \tag{34}$$

yields the following root;

$$r_+ = \frac{0.18D^{\frac{2}{3}} + 0.21aP}{PD^{\frac{1}{3}}}, \quad \text{where} \quad D = \left[a(-1.12b + 0.2\sqrt{\frac{-4a + 27b^2P}{P}})P^2\right].$$

Later, we discuss about heat capacity and phase transition, and compare them with results obtained by other methods.

4 New critical points

Regarding the review of ordinary thermodynamic systems, it is evident that all of the complete differentiations can be written as a function of three thermodynamic coordinates.

These three coordinates are not independent, for example, in most cases, thermodynamic systems can be written in terms of pressure, temperature, internal energy, and free energy of Gibbs, which are independent of each other. We want to get relationships that are independent of each other, but these new relationships must satisfy conditions related to the thermodynamic behavior of the system, such as phase transition and critical behavior. So, instead of using the equation of the ordinary state (which is temperature dependent), we obtain the slope of the temperature in terms of entropy. This equation gives us a new relationship that involves pressure, which only dependent on volume.
Now, we want to present a new method to present these relations for different thermodynamic variables. These new relationships provide the conditions for the system phase transition. It should be noted that in order to obtain a new relationship for pressure, one can use $\left(\frac{\partial^2 H}{\partial s^2}\right)$ instead of $\frac{\partial T}{\partial S}$, where H is an enthalpy of the system.

Now, we calculate the volume conjugating to the pressure,

$$V=\left(\frac{\partial H}{\partial P}\right)_s=\left(\frac{\partial M}{\partial P}\right)_s=\frac{4\pi}{3}r_+^3-\pi b r_+^2, \tag{35}$$

where the black hole mass is considered as the black hole enthalpy [45]. Thus, we are in a position to use the new method. Since both entropy $S(\nu)$ and enthalpy $H(\nu)$ are volume dependent, we can use the following relation [36],

$$\left(\frac{\partial H}{\partial S}\right)_Q=a\frac{\sqrt{\pi}}{2}S^{\frac{-1}{2}}-\frac{\pi ab}{4S}+2\frac{P\sqrt{S}}{\sqrt{\pi}}-Pb \tag{36}$$

and,

$$\left(\frac{\partial^2 H}{\partial S^2}\right)_Q=\frac{1}{S}\left(-\frac{a\sqrt{\pi}}{4}S^{\frac{-1}{2}}+\frac{\pi ab}{4S}+\frac{P\sqrt{S}}{\sqrt{\pi}}\right). \tag{37}$$

In order to solve this relation with respect to P, one can find the following new relation for pressure which differs from the equation of state,

$$P_{new}=\frac{a}{16r_+^2}-\frac{ab}{32r_+^3}. \tag{38}$$

Using the concept of extremum of this relation being the critical point, the critical volume and pressure are given by,

$$\upsilon_c=3b\ ,\quad P_c=\frac{a}{54b^2}. \tag{39}$$

Regarding this relation and replacing corresponding pressure in the temperature (23), mass (26), and Gibbs free energy (29), we obtain the new relations as follows;

$$T_{new} = \frac{a}{r_+}-\frac{ab}{r_+^2}+\frac{ab^2}{4r_+^3}, \tag{40}$$

$$M_{new} = \frac{4}{3}\pi a r_+-\frac{\pi ab}{2}\ln r_+-\frac{7}{12}\pi ab+\frac{\pi ab^2}{4r_+}, \tag{41}$$

$$G_{new} = \frac{23}{12}\pi ab-\frac{5}{3}\pi a r_+-\frac{\pi ab}{2}\ln r_+-\frac{\pi ab^2}{4r_+}. \tag{42}$$

With the new Gibbs free energy, we are able to consider the new stability condition for the corresponding system. We observe from Eq. (42) equation that for the positive values of r_+ the Gibbs free energy has global stability.

The new heat capacity is given by

$$C_{P\ new} = \left(\frac{2S}{3b}\right)\frac{4S - 3b\sqrt{\pi}S^{\frac{1}{2}} - 3\pi b^2}{\sqrt{\pi}S^{\frac{1}{2}} + 2\pi b}. \tag{43}$$

The second order phase transition is governed by the following equation,

$$4S - 3b\sqrt{\pi}S^{\frac{1}{2}} - 3\pi b^2 = 16r_+^2 - 6br_+ - 3b^2 = 0. \tag{44}$$

Eq. (44) is quadratic in nature, thus we can have two possible solutions for r_+;

$$r_{\pm,NC} = \frac{3 \pm \sqrt{57}}{16}b. \tag{45}$$

These solutions are related to the stability of black holes. Thus, the black hole is stable when the horizon radius is $\frac{3+\sqrt{57}}{16}b$ and it is unstable when the horizon radius is $\frac{3-\sqrt{57}}{16}b$. As we said earlier, the second order phase transition is associated with divergence points of the specific heat. Thus we can set the second order partial derivative of M to zero; $\frac{\partial^2 M}{\partial S^2} = 0$. Therefore, we obtain the following,

$$r_+ = b. \tag{46}$$

By analyzing the heat capacity from Eq. (32), we observe that the heat capacity decreases with the increase in the parameter b, thus leading the black hole towards an unstable state. We can also observe that the phase transition occurs for different values of b, and the heat capacity has a stable state for all values of a. For a black hole having a small radius, the different values of parameter a do not affect the heat capacity appreciably. However, for black holes having a considerable radius, increasing a leads to an increase in heat capacity. We also find that the new heat capacity in contrast to the usual case is unstable. Also one can easily obtain the new Helmholtz free energy F_{new} by the following relation:

$$F_{new} = G_{new} - P_{new}V \tag{47}$$

Plugging the values of G_{new}, P_{new} and V from Eq. (40), Eq. (38) and Eq. (35) into Eq. (47) respectively. With some little algebra, we get

$$F_{new} = \frac{93\pi ab}{48} - \frac{21\pi ar_+}{12} - \frac{9\pi ab^2}{32r_+} - \frac{\pi ab}{2}\ln r_+. \tag{48}$$

One can also study the quantum work distribution for such AdS black hole, as it evaporates between two micro-states from Ω_1 to Ω_2. Then the partition function of the black hole will change from Z_1 to Z_2. This will change different thermodynamic quantities between these two states. Now let us assume that an AdS black hole with a partition function $Z_1[\Omega_1]$ evaporates to an AdS black hole with a partition function $Z_2[\Omega_2]$. The term $\frac{Z_2}{Z_1}$ can be related to the average of the exponential of quantum work W, using the Jarzynski equality [52, 53],

$$\left\langle \exp^{-\beta W} \right\rangle = \frac{Z_2}{Z_1}. \tag{49}$$

5 Conclusion

In this paper, we studied the black hole in the Horndeski gravity by studying its $P-V$ behavior. We showed that the usual Horndeski black holes do not have $P-V$ critical behavior. The phase transition leads us, to apply some ansatz which gives us the modified new metric function $f(r)$. This new metric can be used to investigate the critical point for the P_c, T_c, and V_c, and the obtained results are the same as Van der Waals fluid. So, we have shown that the modified Horndeski black hole is satisfied by the equation of state "liquid- gas" phase transition. Our main goal of this paper is to study the $P-V$ criticality behavior of the modified Horndeski black hole. First, we obtained the modified Horndeski black hole by the new definition. We showed that there exists a critical point for $f(r_+)$ in the corresponding black hole which decreases as long as M (mass of the black hole) increases (see Fig. 1).

In order to understand the details of the behavior of the physical mass, we draw the corresponding diagrams in Fig. Here, we pointed out that the values of physical mass first decrease before the horizon radius reaches the critical horizon radius and then it increases when the critical horizon radius is exceeded. We showed that there exists a minimum point for physical mass which decreases as long as the coefficients of VdW fluid increases. We fix the coefficients a and P which correspond to a Van der Waals fluid. As the parameter b increases, the critical point of the physical mass is shifted to the right. In this case, we can call b as a correction quantity. One way to check the stability of the system is by calculation of Gibbs free energy. We found that the coefficients of the VdW increase the stability regions of the black hole when the radius is very small. But the Gibbs free energy completely lies within the state of global stability when the radius is large. Another way to study the stability of the system is to calculate the heat capacity. Using heat capacity, we found that the coefficient b reduces the stable regions of the black hole. But, increasing the coefficient a does not have much effect on the heat capacity. The heat capacity can reach to stable state when the horizon radius is large.

Finally, we applied a new method to study phase transition points in this black hole. In the usual method, the phase transition study originates from the temperature related to the state equation, but the new method is based on the temperature gradient slope of entropy. This new method is a complete method for studying the critical behavior of a thermodynamic system. The results of the new method are similar to other methods, but they provide more information on the critical behavior of thermodynamic systems that we cannot extract through other methods. Also, the analytical interpretation of possible phase transition points leads us to arrange some nonphysical range of horizon radius for the corresponding black hole. Another advantage of this method is that it discusses all thermodynamic quantities. It will be interesting to analyze the effect of the thermal fluctuations on modified black holes on via this new method, and compare its results with the usual method [46, 47, 48, 49].

Authors' contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, and other related matters.

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