Journal of Holography Applications in Physics
Volume 2, Issue 4, Autumn 2022, 11-36
©Available online at http://jhap.du.ac.ir
DOI:10.22128/jhap.2022.584.1032

Journal of Holography Applications in Physics

Regular article

Gauge Invariant Degeneracies and Rotational Symmetry Eigenstates in Noncommutative Plane

M.N.N.M. Rusli [1] · **M.S. Nurisya** [2] · **H. Zainuddin** [3] · **M.F. Umar** [4] · **A. Jellal** [5]

[1] Department of Physics, Kulliyyah of Science (KOS), International Islamic University Malaysia (IIUM), 25200 Kuantan, Pahang, Malaysia.

[2] Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.
and
Department of Physics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.
email: risya@upm.edu.my

[3] Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.
and
Department of Physics, Faculty of Science, Universiti Putra Malaysia, 43400 UPM Serdang, Selangor, Malaysia.

[4] Faculty of Science and Mathematics, Universiti Pendidikan Sultan Idris, 35900 Tanjung Malim, Perak, Malaysia.

[5] Laboratory of Theoretical Physics, Faculty of Sciences, Chouaïb Doukkali University, PO Box 20, 24000 El Jadida, Morocco.
and
Canadian Quantum Research Center, 204-3002 32 Ave Vernon, BC V1T 2L7, Canada.
Corresponding author email: a.jellal@ucd.ac.ma

Received: August 10, 2022; Accepted: September 24, 2022.

Abstract. We calculate the gauge invariant energy eigenvalues and degeneracies of a spinless charged particle confined in a circular harmonic potential under the influence of a perpendicular magnetic field B on a 2D noncommutative plane. The phase space coordinates transformation based on the 2-parameter family of unitarily equivalent irreducible representations of the nilpotent Lie group G_{NC} was used to accomplish this. We find that the energy eigenvalues and quantum states of the system are unique since they depend on the particle of interest and the applied magnetic field B. Without B, we essentially have a noncommutative planar harmonic oscillator under the Bopp shift formulation. The corresponding degeneracy is not unique with respect to the choice of particle, and they are only reliant on the two free integral parameters. The degeneracy is not unique for the scale $B\theta = \hbar$ and is in fact isomorphic to the Landau problem in symmetric gauge; thus, each energy level is infinitely degenerate for any arbitrary magnitude of magnetic field. If $0 < B\theta < \hbar$, the degeneracy is unique with respect to both the particle of interest and the applied magnetic field. The system is, in principle, highly non-degenerate and, in practice, effectively non-degenerate, as only the finely-tuned magnetic field can produce degenerate states. In addition, the degeneracy also depends on the two free integral parameters. Numerical examples are provided to present the degeneracies, probability densities, and effects of B and θ on the ground and excited states of the system for all cases using the physical constants from the numerical simulation and experiment on a single GaAs parabolic quantum dot.

Keywords: Particle on the Noncommutative Plane; Magnetic Field; Harmonic Potential; Gauge Invariance; Degeneracy; Wavefunction; Quantum Dot.

1 Introduction

Noncommutative structure in spacetime coordinates predated Heisenberg and was formally realized by Snyder in 1947 [1] as an effective way to avoid or at least ameliorate the short-distance singularities that plagued quantum field theory and, in particular, gauge theories in the early days. However, this idea only became a subject of interest for a short while due to the remarkable success of the ensuing emergence of the renormalization scheme. We refer to [2–4] for more information on the historical context and relevant reviews. Nonetheless, in recent years, research involving these noncommuting coordinates has gained back its momentum due to the discovery of its applicability in the framework of superstring theories and of quantum gravity [5]. For some reviews of these topics and, in particular, quantum field theory in noncommutative spacetime, we refer to [2, 6].

Numerous works address the low-energy limit of the one-particle sector of noncommutative field theories, i.e., noncommutative quantum mechanics (NCQM), in various settings. This includes harmonic oscillators [7–10], magnetic field [11–15], hydrogen atom [16], central potential [17, 18], Landau problem [19, 20], Klein-Gordon and Dirac oscillators [21–23], Aharonov-Casher effect [24], and so on. One of the authors studied a system of spinless electrons moving in a $2D$ noncommutative space in the presence of a perpendicular magnetic field and confining harmonic potential [25]. His focus was on the orbital magnetism of the electrons in different temperature regimes, magnetic field, and noncommutative parameter θ. In fact, he proved that the degeneracy of Landau levels could be lifted by the θ-term appearing in the electron energy spectrum at a weak magnetic field.

In the above mentioned works, the center of interest usually revolves around eigenvalue problems, which most of the time are treated using an algebraic method, while the analytical approach receives relatively less attention in comparison, which hinders us from gaining direct physical insight into the behavior of the probability densities. Apart from that, the sources which discuss the degeneracies of NCQM models are also noticeably scarce and face the issue of gauge dependency (e.g., in [26, 27]).

As a result, these issues will be our primary focus in this paper, particularly for the Landau problem confined to isotropic harmonic potential in 2D noncommutative phase space. This work does not consider the minimal coupling prescription as is done naively on many occasions in the literature, as it yields gauge dependence of the underlying energy spectra, e.g., in anisotropic harmonic oscillator and quantum Hall effect [11, 14, 28].

Instead, as the kinematical symmetry group of 2D noncommutative quantum mechanics whose Lie algebra has been established in [29] that can produce gauge invariant spectra, we use the families of self-adjoint irreducible representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{NC})$ of the Lie algebra $\mathfrak{g}_{NC}$. In addition, it has been shown in [18] that noncommutative quantum mechanics in 2D central field and Landau problem are equivalent theories under certain conditions, and the authors' earlier work in [10] showed how this happened in harmonic oscillator under less restrictive conditions. It would be interesting to ponder if such a convenient isomorphic model can be derived here despite the inclusion of a mathematical scheme for gauge invariance, at least in the context of degeneracies.

We emphasize that our study focuses on the Fock Darwin system but in gauge invariant noncommutative phase space. The regular commutative Fock Darwin system has been found to be an important elementary system when studying quantum Hall effect [30], and quantum dots [31]. Consequently, we will be using the typical physical constants that have been used in numerical simulations and experiments involving a mesoscopic system i.e., single GaAs parabolic quantum dot [32, 33], which is suitable as an approximate (confining potential) model. The reason for taking momentum noncommutativity comes from the fact that in quantum mechanics, the generalized momentum components are noncommutative. Moreover, the work in [34] has demonstrated that in order to keep the BoseEinstein statistics for identical particles intact at the noncommutative level, we should consider both spacespace and momentummomentum noncommutativity. Related cross-disciplinary research between mathematical and condensed matter physics has also been actively studied due to phenomenological and pedagogical interests in understanding noncommutative effects at mesoscopic scales because numerical simulations are usually implemented at these scales and it is also very difficult to implement physical experiments at the Planck scale directly. Some of the related works are as follows: quantum dot with spin-orbit interaction [35], quantum ring [36], degenerate electron gas [37], superconductor [38], Fermi gas [39], etc.

The paper is organized as follows: In Section 2, we briefly revisit the minimal coupling prescription and state the alternative gauge invariant phase space coordinate transformation to be used later. Then, the time-independent Schrödinger equation of the Landau problem confined in an isotropic harmonic potential on a noncommutative plane is solved to express the energy eigenvalues and eigenstates that emerge. As the coordinate transformation, we use the symmetric part of the 2-parameter (r, s) family of irreducible self-adjoint representations $\mathcal{U}(\mathfrak{g}_{NC})$. It turns out that the product of the system's two parameters, magnetic field and noncommutativity, must be $0 \leq B\theta \leq \hbar$ in order for the effective mass that characterizes the system to be real, which gives rise to three possible cases. As a result, the third section is dedicated to the study of the first case, i.e., in the absence of a magnetic field ($B\theta = 0$). The gauge invariant energy eigenvalues and degeneracies are derived, and the rotationally symmetric eigenstates are defined. We also provide numerical examples to demonstrate the degeneracy. A similar discussion is done in the fourth section as in the previous one for the remaining two cases in the presence of a homogeneous magnetic field for $B\theta = \hbar$ and $0 < B\theta < \hbar$, respectively. The final section is allocated to portray the behavior of the probability density functions at varying quantum number pairs, strengths of a magnetic field, and the noncommutativity of the system for all cases.

2 Energy spectrum on noncommutative plane

Before we begin our discussion on the gauge invariant coordinates transformation, we will briefly revisit the so-called minimal coupling prescription, which is used in much of the literature. The quantum phase space coordinates in standard quantum mechanics are made up of the Hermitian operators $\hat{x}$,$\hat{y}$,$\hat{p}_x$, and $hatp_y$ defined on $L^2(\mathbb{R}^2, dxdy)$ with the following commutation relations

$$[\hat{x}, \hat{y}] = [\hat{p}_x, \hat{p}_y] = 0, \qquad [\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hbar\mathbb{I}, \qquad [\hat{x}, \hat{p}_y] = [\hat{y}, \hat{p}_x] = 0, \tag{2.1}$$

where $\mathbb{I}$ is the identity operator on $L^2\left(\mathbb{R}^2, dr_1 dr_2\right)$. The above commutation relations correspond to the 5-dimensional Weyl-Heisenberg group, G_{WH}. The minimal coupling prescription is derived from the following gauge potential

$$\hat{\mathbf{A}}(\hat{X}, \hat{Y}) = (-B(1-r)\hat{Y}, rB\hat{X}), \tag{2.2}$$

such that Landau and symmetric gauges correspond to $r = 1$ and $r = \frac{1}{2}$ respectively. Later, this prescription can be used naively to write down the kinematical momentum operators as follows

$$\hat{\Pi}_i = \hat{p}_i - e\hat{A}_i, \qquad i = x, y. \tag{2.3}$$

Remember that the Hamiltonian of a charged particle of effective mass m in an external parabolic confinement with an angular frequency of ω in a uniform external magnetic field is denoted by

$$\hat{H} = \frac{1}{2m}\left[\hat{\Pi}_x^2 + \hat{\Pi}_y^2\right] + \frac{1}{2}m\omega^2\left[\hat{x}^2 + \hat{y}^2\right]. \tag{2.4}$$

The above Hamiltonian is sometimes known as the Fock-Darwin Hamiltonian for single-particle attributed to the pioneering authors in [40,41] which gives rise to Fock-Darwin levels and Fock-Darwin states where the Zeeman spin splitting in the magnetic field is neglected. The choice of the above potential to confine the particle works as a good approximation of a confinement scheme in semiconductor quantum dots [31] and quantum Hall effect [30]. In this work, we can say that we are trying to study the same Fock-Darwin system but it is now modified such that it will be treated in 2D noncommutative phase space in which the phase space coordinates transformation used to map from noncommutative to commutative space also gauge invariant.

In literature, there is frequent usage of a particular mapping which is known as the Bopp shift transformation

$$\hat{X} = \hat{x} - \frac{\theta}{2\hbar}\hat{p}_y, \qquad \hat{Y} = \hat{y} + \frac{\theta}{2\hbar}\hat{p}_x, \tag{2.5}$$

$$\hat{P}_x = \hat{p}_x, \qquad \hat{P}_y = \hat{p}_y. \tag{2.6}$$

where $[\hat{X}, \hat{Y}] = i\theta\mathbb{I}$ and θ is a small, positive parameter which measures the additionally introduced noncommutativity between the observables of the two spatial coordinates. In [28], this noncommutative setup has been shown explicitly to yield gauge dependency via eigenfrequencies of the underlying energy spectra for the cases of anisotropic harmonic oscillator and quantum Hall effect, which is inconsistent in the context of noncommutative quantum mechanics. As a result, in this paper, we will rely on families of self-adjoint irreducible representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{NC})$ of the Lie algebra $\mathfrak{g}_{NC}$, the kinematical symmetry group of 2D NCQM, whose corresponding Lie group G_{NC} has been established in an earlier paper [29]. We will not delve any further into the theoretical group structure as that has been done in [28]. What we are interested in is using the result of gauge invariant coordinates transformation in the paper, i.e., the 2-parameter family of equivalent self-adjoint irreducible representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{NC})$ on the smooth vectors of $L^2(\mathbb{R}^2, dxdy)$, a family to which Landau and symmetric gauge representations belong, where it is formulated as below

$$\hat{X}^s = \hat{x} - s\frac{\theta}{\hbar}\hat{p}_y, \tag{2.7}$$

$$\hat{Y}^s = \hat{y} + (1-s)\frac{\theta}{\hbar}\hat{p}_x, \tag{2.8}$$

$$\hat{\Pi}_x^{r,s} = \frac{(1-r)\hbar B}{\hbar - r\theta B}\left(\hat{y} - \frac{s\theta}{\hbar}\hat{p}_x\right) + \hat{p}_x, \tag{2.9}$$

$$\hat{\Pi}_y^{r,s} = -rB\left[\hat{x} + \frac{(1-s)\theta}{\hbar}\hat{p}_y\right] + \hat{p}_y, \tag{2.10}$$

where the Landau and symmetric gauges correspondent to, respectively,

$$r = 1, \quad s = 0, \tag{2.11}$$

$$r = \frac{\hbar}{\hbar + \sqrt{\hbar^2 - \hbar B\theta}}, \qquad s = \frac{1}{2}. \tag{2.12}$$

Hence, the Fock-Darwin Hamitonian in gauge invariant 2D noncommutative phase space can be written as

$$\hat{H} = \frac{1}{2m}\left[\left(\hat{\Pi}_x^{r,s}\right)^2 + \left(\hat{\Pi}_y^{r,s}\right)^2\right] + \frac{1}{2}m\omega^2\left[\left(\hat{X}^s\right)^2 + \left(\hat{Y}^s\right)^2\right]. \tag{2.13}$$

In this work, we will be focusing on the symmetric gauge part since the energy eigenvalues and the associated degeneracies that will be obtained can naturally be extended to Landau gauge as well due to gauge invariance. For the wavefunctions, the mathematical structure should only apply to the symmetric gauge and also to any other gauge choices with underlying rotational symmetry under simple substitution. The corresponding gauge invariant (symmetric gauge part) phase space coordinate transformations between noncommutative operators and commutative ones based on a 2-parameter family of an equivalent self-adjoint irreducible representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{NC})$ on the smooth vectors of $L_2(\mathbb{R}^2, dxdy)$ is [28]

$$\hat{X} = \hat{x} - \frac{\theta}{2\hbar}\hat{p}_y, \tag{2.14}$$

$$\hat{Y} = \hat{y} + \frac{\theta}{2\hbar}\hat{p}_x, \tag{2.15}$$

$$\hat{\Pi}_x = \frac{\hbar B}{\hbar + \sqrt{\hbar(\hbar - B\theta)}}\hat{y} + \frac{\hbar + \sqrt{\hbar(\hbar - B\theta)}}{2\hbar}\hat{p}_x, \tag{2.16}$$

$$\hat{\Pi}_y = -\frac{\hbar B}{\hbar + \sqrt{\hbar(\hbar - B\theta)}}\hat{x} + \frac{\hbar + \sqrt{\hbar(\hbar - B\theta)}}{2\hbar}\hat{p}_y. \tag{2.17}$$

The self-adjoint differential operators on the space of smooth vectors of $L^2(\mathbb{R}^2)$ obey the following commutation relations

$$\left[\hat{X}, \hat{Y}\right] = i\theta\mathbb{I}, \tag{2.18}$$

$$\left[\hat{\Pi}_x, \hat{\Pi}_y\right] = i\hbar B\mathbb{I}, \tag{2.19}$$

$$\left[\hat{X}, \hat{\Pi}_x\right] = \left[\hat{Y}, \hat{\Pi}_y\right] = i\hbar\mathbb{I}, \tag{2.20}$$

$$\left[\hat{X}, \hat{\Pi}_y\right] = \left[\hat{Y}, \hat{\Pi}_x\right] = 0, \tag{2.21}$$

where $\mathbb{I}$ being the identity operator on $L^2\left(\mathbb{R}^2, dr_1 dr_2\right)$. Note that the magnetic field, i.e., B can be rescaled to $B \to \dfrac{eB}{c}$ to connect our notation with the usual literature on the Landau problem. In [28], the cyclotron frequency was defined to be $\omega_c = \frac{B}{m}$. Strictly speaking, in SI units, due to the absence of the charge of a particle of interest in ω, we can consider that it is already being absorbed in B. Therefore, throughout this work, B will be defined as the magnetic field multiplied by a unit charge. However, we will simply call it magnetic field in the sequel.

By substituting from (2.14) until (2.17) into (2.13), the Hamiltonian is of the form

$$\begin{aligned}\hat{H} =& \frac{1}{2m}\left(\frac{\hbar B}{\hbar+\sqrt{\hbar(\hbar-B\theta)}}\hat{y}+\frac{\hbar+\sqrt{\hbar(\hbar-B\theta)}}{2\hbar}\hat{p}_x\right)^2 \\ &+\frac{1}{2m}\left(-\frac{\hbar B}{\hbar+\sqrt{\hbar(\hbar-B\theta)}}\hat{x}+\frac{\hbar+\sqrt{\hbar(\hbar-B\theta)}}{2\hbar}\hat{p}_y\right)^2 \\ &+\frac{1}{2}m\omega^2\left(\hat{x}-\frac{\theta}{2\hbar}\hat{p}_y\right)^2+\frac{1}{2}m\omega^2\left(\hat{y}+\frac{\theta}{2\hbar}\hat{p}_x\right)^2.\end{aligned} \tag{2.22}$$

After a few algebraic manipulation steps, we obtain

$$\begin{aligned}\hat{H} =& \left(\frac{2\hbar^2+2\hbar\sqrt{\hbar^2-\hbar B\theta}-\hbar B\theta+m^2\omega^2\theta^2}{8m\hbar^2}\right)\left(\hat{p}_x^2+\hat{p}_y^2\right) \\ &+\left(\frac{\hbar^2B^2+m^2\omega^2(2\hbar^2+2\hbar\sqrt{\hbar^2-\hbar B\theta}-\hbar B\theta)}{2m(2\hbar^2+2\hbar\sqrt{\hbar^2-\hbar B\theta}-\hbar B\theta)}\right)\left(\hat{x}^2+\hat{y}^2\right)-\left(\frac{\hbar B+m^2\omega^2\theta}{2\hbar m}\right)\hat{L}_z\end{aligned} \tag{2.23}$$

where $\hat{L}_z$ is the z-component of the projection of the angular momentum onto the field direction. By rearranging (2.23), we can introduce a new effective mass and frequency such that the expression above excluding the $\hat{L}_z$-term can have the form of the Hamiltonian of a planar isotropic harmonic oscillator as follows

$$\hat{H} = \frac{1}{2M}\left(\hat{p}_x^2+\hat{p}_y^2\right)+\frac{1}{2}M\Omega^2\left(\hat{x}^2+\hat{y}^2\right)-\gamma\hat{L}_z. \tag{2.24}$$

Hence, by comparing (2.23) and (2.24), the effective mass is

$$M = \frac{4m\hbar^2}{2\hbar^2+2\hbar\sqrt{\hbar^2-\hbar B\theta}-\hbar B\theta+m^2\omega^2\theta^2}, \tag{2.25}$$

whereas the effective frequency is calculated by comparing (2.23), (2.24) and (2.25)

$$\Omega = \sqrt{\omega^2+\frac{\left(\hbar B-m^2\omega^2\theta\right)^2}{4\hbar^2m^2}} \tag{2.26}$$

and we have set γ as

$$\gamma = \frac{\hbar B+m^2\omega^2\theta}{2\hbar m}. \tag{2.27}$$

Since the Hamiltonian (2.24) is rotationally symmetric, it is appropriate to work in polar coordinates (r,φ). Then the stationary Schrödinger equation is

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2\Psi(r,\varphi)}{\partial r^2}+\frac{1}{r}\frac{\partial\Psi(r,\varphi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Psi(r,\varphi)}{\partial\varphi^2}\right)+\frac{1}{2}M\Omega^2r^2\Psi(r,\varphi)-\gamma L_z\Psi(r,\varphi) = E\Psi(r,\varphi), \tag{2.28}$$

where $L_z = -i\hbar\frac{\partial}{\partial\varphi}$. The solutions of the energy spectrum give the eigenvalues

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\,\hbar\Omega - m_l\hbar\gamma, \tag{2.29}$$

as well as the associated wavefunctions

$$\Psi_{n_r,m_l}(r,\varphi) = \frac{1}{\sqrt{2\pi r}} R_{n_r,m_l}(r) e^{im_l\varphi}, \tag{2.30}$$

in which $R_{n_r,l}(r)$ is the radial part

$$R_{n_r,m_l}(r) = \left(\frac{2M\Omega}{\hbar}\right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\sqrt{r}\left(\frac{M\Omega}{\hbar}r^2\right)^{\frac{|m_l|}{2}} \exp\left(-\frac{M\Omega}{2\hbar}r^2\right) L_{n_r}^{|m_l|}\left(\frac{M\Omega}{\hbar}r^2\right), \tag{2.31}$$

and $L_{n_r}^{m_l}$ is the Laguerre polynomials [9, 42], $n_r \in \mathbb{N}$ and $m_l \in \mathbb{Z}$ are radial and angular momentum quantum numbers respectively.

Realize that there is a condition to be satisfied for the solution of the eigenvalue problem before it can really be applied to a physical system. In (2.25), the expression denotes the effective mass of a particle in the oscillator potential, which is real and greater than 0. That is

$$\hbar^2 - \hbar B\theta \geqslant 0. \tag{2.32}$$

Because we can manipulate the magnitude of the magnetic field, it must be non-negative and real, i.e., $B \geqslant 0$. For the noncommutativity parameter, it is also non-negative and real. However, we are not interested in analyzing the situation at $\theta = 0$ since we are discussing the NCQM model. Consequently, we have

$$0 \leqslant B\theta \leqslant \hbar. \tag{2.33}$$

Due to the above constraint, we will analyze in greater detail the three possible cases in the upcoming sections, i.e.,

$$\begin{aligned} \text{Case I:} &\quad B\theta = 0, \\ \text{Case II:} &\quad B\theta = \hbar, \\ \text{Case III:} &\quad 0 < B\theta < \hbar. \end{aligned} \tag{2.34}$$

3 Absence of B

3.1 Case I: $B\theta = 0$

In the absence of a magnetic field, the solution of the eigenvalue problem in (2.29) and (2.30) can be simplified as

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\,\hbar\left(\sqrt{\omega^2\left(1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}\right)}\right) - m_l\hbar\left(\frac{m\omega^2\theta}{2\hbar}\right), \tag{3.1}$$

$$\begin{aligned} \Psi^{\theta}_{n_r,m_l}(r,\varphi) =& \frac{1}{\sqrt{2\pi}}\left(\frac{4m\omega}{\sqrt{4\hbar^2+m^2\omega^2\theta^2}}\right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\left(\frac{2m\omega}{\sqrt{4\hbar^2+m^2\omega^2\theta^2}}r^2\right)^{\frac{|m_l|}{2}} \\ & \exp\left(-\frac{m\omega}{\sqrt{4\hbar^2+m^2\omega^2\theta^2}}r^2\right) L_{n_r}^{|m_l|}\left(\frac{2m\omega}{\sqrt{4\hbar^2+m^2\omega^2\theta^2}}r^2\right) e^{im_l\varphi}. \end{aligned} \tag{3.2}$$

A close inspection reveals that the eigenvalues and eigenstates shown in (3.1) and (3.2) are quite familiar in some literature (e.g. [9, 25]) as they are actually the solution of the eigenvalue problem involving noncommutative planar isotropic harmonic oscillator if the coordinate transformation used is the Bopp shift formulation. The energy eigenvalues of the system for the first few lower quantum number pairs are shown in Table 1. As can be seen,

$(n_r, 0)$	Energy	$(0, m_l)$	Energy	(n_r, m_l)	Energy
$(1, 0)$	$3\hbar\Omega$	$(0, 1)$	$2\hbar\Omega - \hbar\gamma$	$(1, -1)$	$4\hbar\Omega + \hbar\gamma$
$(2, 0)$	$5\hbar\Omega$	$(0, 2)$	$3\hbar\Omega - 2\hbar\gamma$	$(1, -2)$	$5\hbar\Omega + 2\hbar\gamma$
$(3, 0)$	$7\hbar\Omega$	$(0, 3)$	$4\hbar\Omega - 3\hbar\gamma$	$(2, -1)$	$6\hbar\Omega + \hbar\gamma$
		$(0, -1)$	$2\hbar\Omega + \hbar\gamma$	$(1, 1)$	$4\hbar\Omega - \hbar\gamma$
		$(0, -2)$	$3\hbar\Omega + 2\hbar\gamma$	$(1, 2)$	$5\hbar\Omega - 2\hbar\gamma$
		$(0, -3)$	$4\hbar\Omega + 3\hbar\gamma$	$(2, 1)$	$6\hbar\Omega - \hbar\gamma$

Table 1: Energy eigenvalues of the first few ground and excited states designated by the quantum number pairs (n_r, m_l) in terms of Ω and γ.

it is not immediately obvious to infer if there is any hidden pattern in the distribution of energy. Hence, we will express γ in terms of Ω in (3.1) as this step is crucial in simplifying the analysis later on

$$\gamma = \left(\sqrt{1 - \frac{\omega^2}{\Omega^2}}\right)\Omega = \kappa\Omega, \tag{3.3}$$

where $0 < \kappa < 1$ is a direct consequence since the frequency, ω is nonzero. We can then rewrite the energies as tabulated in Table 2.

$(n_r, 0)$	Energy	$(0, m_l)$	Energy	(n_r, m_l)	Energy
$(1, 0)$	$3\hbar\Omega$	$(0, 1)$	$(2 - \kappa)\hbar\Omega$	$(1, -1)$	$(4 + \kappa)\hbar\Omega$
$(2, 0)$	$5\hbar\Omega$	$(0, 2)$	$(3 - 2\kappa)\hbar\Omega$	$(1, -2)$	$(5 + 2\kappa)\hbar\Omega$
$(3, 0)$	$7\hbar\Omega$	$(0, 3)$	$(4 - 3\kappa)\hbar\Omega$	$(2, -1)$	$(6 + \kappa)\hbar\Omega$
		$(0, -1)$	$(2 + \kappa)\hbar\Omega$	$(1, 1)$	$(4 - \kappa)\hbar\Omega$
		$(0, -2)$	$(3 + 2\kappa)\hbar\Omega$	$(1, 2)$	$(5 - 2\kappa)\hbar\Omega$
		$(0, -3)$	$(4 + 3\kappa)\hbar\Omega$	$(2, 1)$	$(6 - \kappa)\hbar\Omega$

Table 2: Energy eigenvalues of the first few ground and excited states designated by the quantum number pairs (n_r, m_l) in terms of Ω.

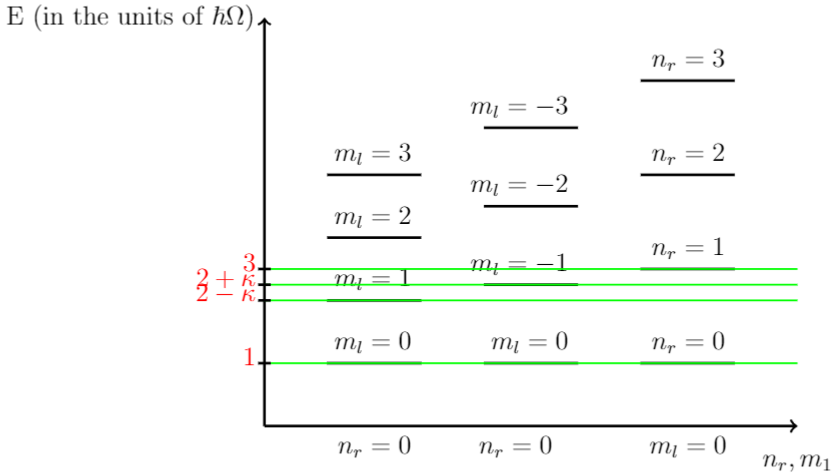

Figure 1: (color online) Energy level diagram for the first few ground and excited states designated by the quantum number pairs $(n_r, 0)$ and $(0, m_l)$.

If one of the two quantum numbers is zero, it will result in the energy spectrum being equidistant from one another, as can be observed more clearly in figure 1. Clearly, we notice that the effect of the individual quantum numbers n_r and m_l differs ever so slightly in the minimal discrete energy step or quanta of energy, δE, depending on κ. The sign of m_l also has an effect on δE

$$\begin{aligned}&\delta E_{m_l>0} < \delta E_{m_l<0} < \delta E_{n_r},\\ &(1-\kappa)\hbar\Omega < (1+\kappa) < 2\hbar\Omega.\end{aligned} \tag{3.4}$$

However, what really matters is the joint effect of both quantum numbers, which has to be determined to see if the system has any degeneracy. If there exist, at least two successive degenerate energy levels can always be determined, such as

$$E_{n_r;1,m_l;1} = E_{n_r;2,m_l;2}. \tag{3.5}$$

Otherwise, we have

$$\begin{aligned}&(2n_{r;1} + |m_{l;1}| + 1)\,\hbar\Omega - m_{l;1}\hbar\kappa\Omega = (2n_{r;2} + |m_{l;2}| + 1)\,\hbar\Omega - m_{l;2}\hbar\kappa\Omega,\\ &\kappa = \frac{2(n_{r;2} - n_{r;1}) + (|m_{l;2}| - |m_{l;1}|)}{m_{l;2} - m_{l;1}}.\end{aligned} \tag{3.6}$$

Consequently, we compare (3.3) and (3.6) to get

$$\kappa = \sqrt{1 - \frac{4\hbar^2}{4\hbar^2 + m^2\omega^2\theta^2}}. \tag{3.7}$$

As long as the above equation is satisfied, the degenerate energy levels can be found. The left-hand side of the equation above is a rational fraction. Then, the corresponding right hand side has to be a rational fraction as well. As a result, it is natural to eliminate the $\hbar^2$-term. This is only possible if the product $m^2\omega^2\theta^2$ is a positive rational number multiple of $\hbar^2$, i.e. $m^2\omega^2\theta^2 = c^2\hbar^2$.

Therefore, we obtain

$$\kappa = \sqrt{\frac{c^2}{4+c^2}}. \tag{3.8}$$

Then, by using the method of induction (where the relevance of the square term in c^2 is in simplifying the method), the sequence of possible values of c^2 can be determined as

$$c_{n,k}^2 = \frac{k^2}{n(n+k)}, \tag{3.9}$$

and we have attached the subscript, i.e., $c_{n,k}$ in the sequel. To check the validity of the above proposed sequence

$$\sqrt{\frac{c_{n,k}^2}{4+c_{n,k}^2}} = \frac{k}{2n+k}, \tag{3.10}$$

which is indeed rational. To sum up, the degenerate energy levels of the system in the case I, i.e., in the absence of a magnetic field, where the system is similar to the noncommutative planar harmonic oscillator under minimal coupling prescription, can always be found if it is of the form

$$E_d = (2n_r + |m_l| + 1)\hbar\Omega - \frac{k}{2n+k} m_l \hbar\Omega. \tag{3.11}$$

The above is true for any particle of interest with any mass and frequency. It is obvious that different masses and frequencies will affect the energy gap of the system. When it comes to the degeneracy of the system, we can at least claim that it is irrespective of the particle's physical parameters. There is, however, no single conclusive statement on the exact degeneracy of the system as it is dependent upon the choice of the free integral parameters n and k. In other words, different values of n and k can lead to different answers depending on the degeneracy of the system. If any selected particle is, say, two-fold degenerate, we can pick any other particle and state that it has the exact same degeneracy as long as the free parameters are identical. This will become obvious when we work on the numerical example in the later section.

Let us now turn our attention to the set of noncommutativity parameter values that can be associated with degenerate energy levels θ_d (for this quantity, we will coin the term "repetitive noncommutativity" and use it throughout the paper). It has been found to be unique. To be more specific, we have to describe it as a function of the mass and frequency of the particle

$$\theta_d = \left\{ \left(\frac{\hbar}{m\omega}\right) c_{n,k} \middle| c_{n,k} = \frac{k}{\sqrt{n(n+k)}}, n,k \in \mathbb{Z} > 0 \right\}. \tag{3.12}$$

The above result is also dimensionally consistent because the noncommutativity is in m^2 units, whereas mass, angular frequency, and reduced Planck constant are in kg, s^{-1} and $\mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-1}$ units, respectively.

The factor $c_{n,k}$ is dimensionless. We also want to note that a similar study for the case I has been done in [26] with a different approach, i.e., through an algebraic method. The difference is that they describe the repetitive noncommutativity in terms of two relatively prime numbers, whereas we describe it in terms of two positive integers. It is important to realize that we do not assert that all noncommutativity has to be a function of mass and frequency, but at least that is the case for repetitive noncommutativity. If the energy eigenvalues and noncommutativity parameters have any random values which do not match with (3.11) and (3.12), respectively, then we say those values are not associated with any degeneracy of the system.

By equating the left-hand side of (3.7) and the right-hand side of (3.10), the three successive degenerate energy levels can be found. When the angular momentum quantum numbers for any two degenerate levels are both positive,

$$E_{n_{r;1},m_{l;1}} = E_{n_{r;1}-n,m_{l;1}+2n+k} = E_{n_{r;1}+n,m_{l;1}-(2n+k)}. \tag{3.13}$$

If, however, we are focusing on both angular momentum quantum numbers being negative, then

$$E_{n_{r;1},-m_{l;1}} = E_{n_{r;1}+n+k,-m_{l;1}+2n+k} = E_{n_{r;1}-(n+k),-m_{l;1}-(2n+k)}. \tag{3.14}$$

We will not discuss the situation when either one of the angular momentum quantum numbers is negative, as we can just compare the energy levels of these two formulas to see their equivalence. As a result, when (3.13) and (3.14) are satisfied, any successive degenerate energy levels for a particular θ and thus excited states of the same energy can always be found. Otherwise, the system is non-degenerate.

3.2 Numerical example

Here, we will consider the parameters to be the ones used in numerical simulations and experiments in regular commutative Fock-Darwin systems as documented in [32, 33] for the approximate model of a single GaAs parabolic quantum dot. The effective mass of the conduction band electron for the chosen material is

$$m_{\text{eff}} = 0.067m_e = 0.067\left(9.109 \times 10^{-31}\right)\text{kg} = 6.103 \times 10^{-32}\,\text{kg}. \tag{3.15}$$

The electrostatic confinement energy is given by

$$\begin{aligned}\hbar\omega_0 &= 3\,\text{meV},\\ \omega_0 &= 4.555 \times 10^{12}\,\text{s}^{-1}.\end{aligned} \tag{3.16}$$

Then, for simplicity, we will let $n = k$. At $n = k$, it does not matter what the actual value of this constant is as it will always produce $c_{n,k} = \dfrac{1}{\sqrt{2}}$. Hence, the degenerate energy levels and repetitive noncommutativity will be

$$E_d = (2n_r + |m_l| + 1)\hbar\Omega - \frac{1}{3}m_l\hbar\Omega, \tag{3.17}$$

and

$$\theta_d = \frac{1}{\sqrt{2}}\left(\frac{\hbar}{m_{\text{eff}}\omega_0}\right) = 2.683 \times 10^{-16}\,\text{m}^2. \tag{3.18}$$

The energy eigenvalues of the case $B\theta = 0$ at $\theta_d = 2.683 \times 10^{-16}\,\text{m}^2$ for the ground and excited states designated by the different quantum number pairs are shown in units of $\hbar\Omega = 3.182\,\text{meV}$, obtained from (3.1), in tables 3 and 4.

Energy (in units of $\hbar\Omega$)	(n_r, m_l)
$25 \times \frac{1}{3}$	$(3, 2), (2, 5), (1, 8), (0, 11)$
$23 \times \frac{1}{3}$	$(3, 1), (2, 4), (1, 7), (0, 10)$
$21 \times \frac{1}{3} = 7$	$(3, 0), (2, 3), (1, 6), (0, 9)$
$19 \times \frac{1}{3}$	$(2, 2), (1, 5), (0, 8)$
$17 \times \frac{1}{3}$	$(2, 1), (1, 4), (0, 7)$
$15 \times \frac{1}{3} = 5$	$(2, 0), (1, 3), (0, 6)$
$13 \times \frac{1}{3}$	$(1, 2), (0, 5)$
$11 \times \frac{1}{3}$	$(1, 1), (0, 4)$
$9 \times \frac{1}{3} = 3$	$(1, 0), (0, 3)$
$7 \times \frac{1}{3}$	$(0, 2)$
$5 \times \frac{1}{3}$	$(0, 1)$
$3 \times \frac{1}{3} = 1$	$(0, 0)$

Table 3: (color online) Ordered pair of quantum numbers and its corresponding energy for $m_l \geq 0$ at $B\theta = 0$ and $\kappa = \frac{1}{3}$.

The tables are highlighted to show that for every $3\times\delta E$, there will be one more degenerate state in the subsequent levels. This pattern holds true for all $m_l \geq 0$ higher-order states. For the remaining case of $m_l < 0$, the pattern in the distribution of degenerate energy levels is not apparent at first glance. However, when we highlight the table to separate the energy levels consisting of even and odd n_r states and treat them separately, we notice that the behavior is similar to what we saw earlier when $m_l \geq 0$. As previously stated, we can freely manipulate the particle parameters in the quantum dot system setup because it will only affect the resulting repetitive noncommutativity and energy gap, but the distribution of degenerate energy levels will remain the same as shown in Tables 3 and 4 as long as the free parameter $c_{n,k}$ is kept constant. It is noteworthy that different choices of $c_{n,k}$ can lead to different ways of distributing the degenerate energy levels as opposed to the ones shown in the tables above. In other words, different degenerate energy levels distributed differently imply different degeneracy.

Energy (in units of $\hbar\Omega$)	(n_r, m_l)
$53 \times \frac{1}{3}$	$(1,-11),(3,-8),(5,-5),(7,-2)$
$51 \times \frac{1}{3} = 17$	$(0,-12),(2,-9),(4,-6),(6,-3)$
$49 \times \frac{1}{3}$	$(1,-10),(3,-7),(5,-4),(7,-1)$
$47 \times \frac{1}{3}$	$(0,-11),(2,-8),(4,-5),(6,-2)$
$45 \times \frac{1}{3} = 15$	$(1,-9),(3,-6),(5,-3)$
$43 \times \frac{1}{3}$	$(0,-10),(2,-7),(4,-4),(6,-1)$
$41 \times \frac{1}{3}$	$(1,-8),(3,-5),(5,-2)$
$39 \times \frac{1}{3} = 13$	$(0,-9),(2,-6),(4,-3)$
$37 \times \frac{1}{3}$	$(1,-7),(3,-4),(5,-1)$
$35 \times \frac{1}{3}$	$(0,-8),(2,-5),(4,-2)$
$33 \times \frac{1}{3} = 11$	$(1,-6),(3,-3)$
$31 \times \frac{1}{3}$	$(0,-7),(2,-4),(4,-1)$
$29 \times \frac{1}{3}$	$(1,-5),(3,-2)$
$27 \times \frac{1}{3} = 9$	$(0,-6),(2,-3)$
$25 \times \frac{1}{3}$	$(1,-4),(3,-1)$
$23 \times \frac{1}{3}$	$(0,-5),(2,-2)$
$21 \times \frac{1}{3} = 7$	$(1,-3)$
$19 \times \frac{1}{3}$	$(0,-4),(2,-1)$
$17 \times \frac{1}{3}$	$(1,-2)$
$15 \times \frac{1}{3} = 5$	$(0,-3)$
$13 \times \frac{1}{3}$	$(1,-1)$
$11 \times \frac{1}{3}$	$(0,-2)$
$7 \times \frac{1}{3}$	$(0,-1)$

Table 4: (color online) Ordered pair of quantum numbers and its corresponding energy for $m_l < 0$ at $B\theta = 0$ and $\kappa = \frac{1}{3}$.

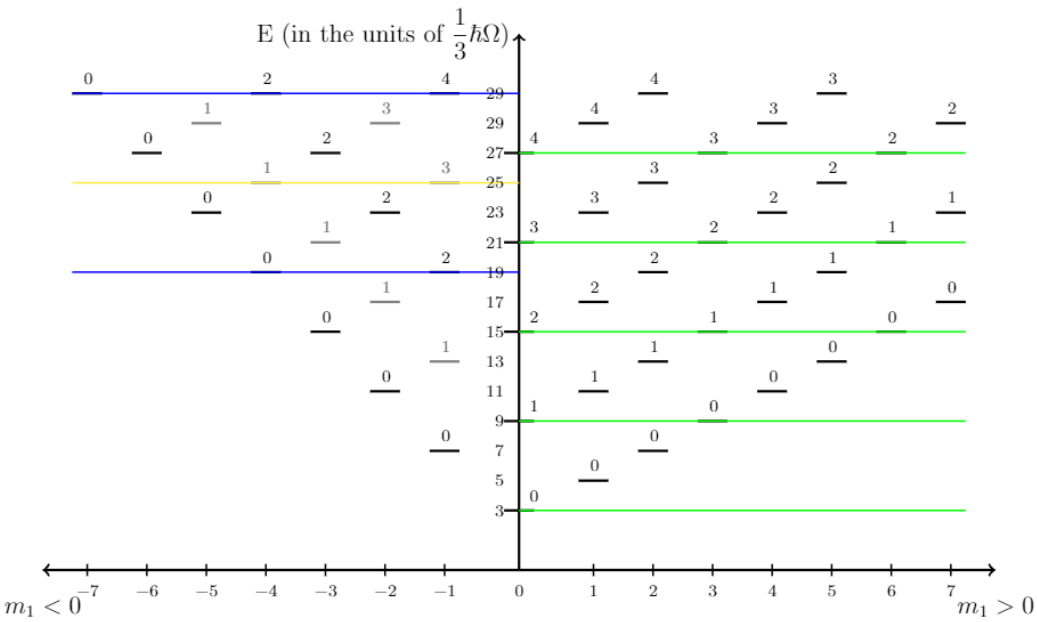

Figure 2: (color online) Energy level diagram of the case $B\theta = 0$ at $\kappa = \frac{1}{3}$.

Figure 2 displays the side-by-side comparison of the asymmetric distribution of energy eigenvalues of positive and negative m_l states. Every integer on the black lines (energy levels) represents a radial quantum number, n_r. Each green lines in figure 2 signifies the level at which we start to have a single additional degenerate state compared to the previous line. The region between them should be occupied by a similar number of degenerate states. The same is true for the negative m_l domain; however, to distinguish between even and odd n_r states, we use blue and yellow lines, respectively. In this case, the system is $(l+1)$-fold degenerate for each half of the quanta of energy $2\delta E$, where l is a positive integer.

4 Presence of B

Based on (2.33)2, noncommutativity should be restricted in the presence of a magnetic field such that the product $B\theta$ is between 0 and $\hbar$. In this section, we will explore the remaining cases, i.e., $B\theta = \hbar$ and $0 < B\theta < \hbar$ respectively.

4.1 Case II: $B\theta = \hbar$

At the extreme end, $B\theta = \hbar$, the solution of the eigenvalue problem in (2.29) and (2.30) can be simplified as

$$E_{n_r,m_l} = (2n_r + |m_l| + 1)\,\hbar\left(\frac{\hbar B + m^2\omega^2\theta}{2\hbar m}\right) - m_l\hbar\left(\frac{\hbar B + m^2\omega^2\theta}{2\hbar m}\right), \tag{4.1}$$

$$\Psi^{B,\theta}_{n_r,m_l}(r,\varphi) = \frac{1}{\sqrt{2\pi}}\left(\frac{2}{\hbar}\left(\frac{2\hbar^2 B + 2hm^2\omega^2\theta}{\hbar^2 + m^2\omega^2\theta^2}\right)\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m_l|)!}}\left(\frac{1}{\hbar}\left(\frac{2\hbar^2 B + 2hm^2\omega^2\theta}{\hbar^2 + m^2\omega^2\theta^2}\right)r^2\right)^{\frac{|m_l|}{2}}$$
$$\exp\left(-\frac{1}{2\hbar}\left(\frac{2\hbar^2 B + 2hm^2\omega^2\theta}{\hbar^2 + m^2\omega^2\theta^2}\right)r^2\right)L^{|m_l|}_{n_r}\left(\frac{1}{\hbar}\left(\frac{2\hbar^2 B + 2hm^2\omega^2\theta}{\hbar^2 + m^2\omega^2\theta^2}\right)r^2\right)e^{im_l\varphi}. \tag{4.2}$$

Recall that in [18], it has been revealed that noncommutative quantum mechanics in 2D central potential and the Landau problem are equivalent under certain conditions. The authors also exhibited in [10] how this is demonstrated in harmonic oscillators under more generalized conditions. Having this isomorphic model is convenient as the Landau problem is a very well understood problem in comparison to noncommutative quantum mechanical models. Because the preceding results yield $\Omega = \gamma$, this isomorphic feature can also be seen in this case, and the degeneracy for the energy eigenvalues in (4.1) is actually equivalent to that in the Landau problem in symmetric gauge. Since the degeneracy of the Landau problem in symmetric gauge is very well known, we will not provide any numerical examples as in the cases I and III.

Every energy level of the Landau problem in symmetric gauge is infinitely degenerate and hence, this applies to this case. As a refresher, the distribution of degenerate energy levels is shown in figure 3 below. As can be seen, the infinite degeneracy in this setting is asymmetrical with respect to the sign of m_l just like in the first case. The electron appears to prefer a direction of L_z in response to the magnetic field and noncommutativity at $B\theta = \hbar$, requiring more energy to exist in the positive states than the negative states. In this noncommutative space, it costs energy for an electron circulating with positive angular momentum, while it is not the case when circulating with negative angular momentum [43]. We also want to emphasize an important point, and that is that in the case II, the magnetic field can always be chosen arbitrarily and the resulting noncommutativity can be evaluated from the said relation, i.e., $B\theta = \hbar$. These values of noncommutativity parameters are all associated with the infinitely degenerate energy levels and, hence, they are all repetitive noncommutativity, i.e.,

$$\theta_d = \theta. \tag{4.3}$$

The degeneracy here has some similar and different characteristics in comparison to the case I. It is similar to the extent that it is also irrespective of the choice of a particle of interest, as the particle's parameters only affect the energy gap. The difference is that the repetitive noncommutativity is not unique to any particle, as it will always follow $B\theta = \hbar$. Apart from that, the degeneracy and hence the distribution of degenerate energy levels can only take the above configuration in figure 3 for any particle and any applied magnetic field. This is to be expected as the degeneracy here does not depend on any free parameter as in the case I.

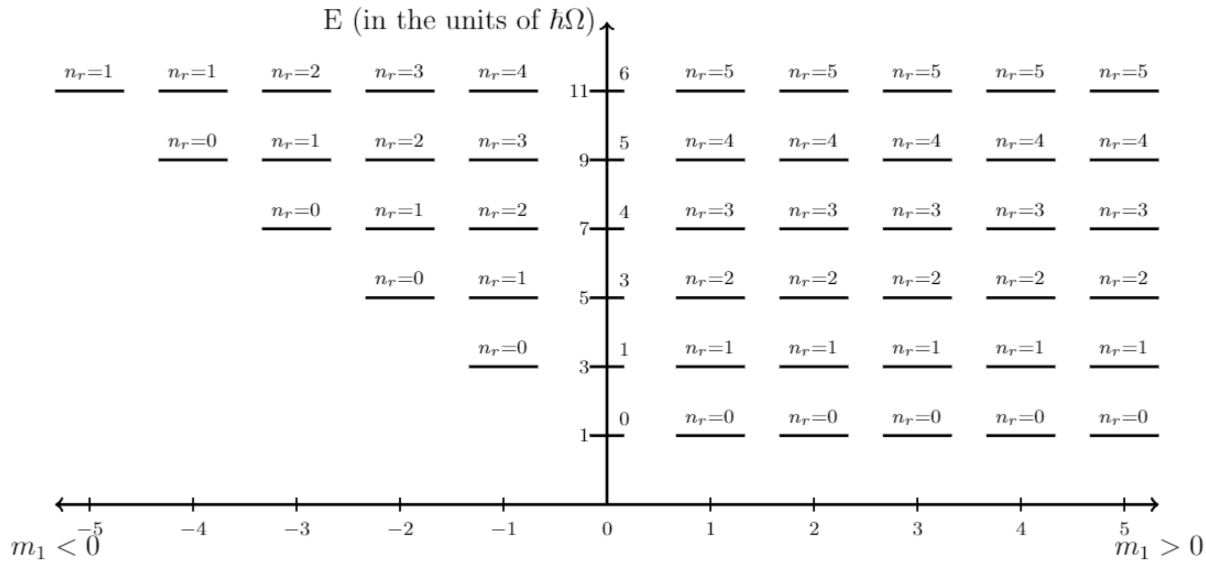

Figure 3: Energy level diagram of the case $B\theta = \hbar$ for $m_l < 0$.

4.2 Case III: $0 < B\theta < \hbar$

The solution to the eigenvalue problem in (2.29) and (2.30) cannot be simplified further, so we will use the explicit form of M, Ω and γ in the following analysis. By squaring and rearranging (2.26) and (2.27) will enable us to spot the common and distinct terms

$$\Omega^2 = \frac{B^2}{4m^2} + \left[\omega^2 - \frac{\omega^2 B\theta}{2\hbar}\right] + \frac{m^2\omega^4\theta^2}{4\hbar^2}, \tag{4.4}$$

$$\gamma^2 = \frac{B^2}{4m^2} + \left[\frac{\omega^2 B\theta}{2\hbar}\right] + \frac{m^2\omega^4\theta^2}{4\hbar^2}. \tag{4.5}$$

After a few algebraic manipulation steps, we can then express γ in terms of Ω as follows, just like in the case I,

$$\gamma = \left(\sqrt{1 + \frac{\omega^2}{\Omega^2}\left(\frac{B\theta}{\hbar} - 1\right)}\right)\Omega = \xi\Omega. \tag{4.6}$$

A simple check reveals that the domain of ξ is identical to that of κ, i.e., $0 < \xi < 1$. From this point, the discussion and arguments that lead to degeneracy are very much similar to the case I and hence will not be repeated here. We are specifically referring to the effect of individual quantum numbers, as shown in figure 2, and ξ is equivalent to κ in terms of the quotient of quantum number difference, as shown in (3.6). Later on, we will let the magnetic field and noncommutativity be expressed as

$$B = fm\omega, \qquad \theta = g\frac{\hbar}{m\omega}, \tag{4.7}$$

as they will rationalize the expression of ξ in (4.6) Furthermore, θ, like the case I, is dimensionally consistent. If we consider B to be the magnetic field, it is supposed to be in the units of $\mathrm{kg} \cdot \mathrm{A}^{-1} \cdot \mathrm{s}^{-2}$, but since B is really the magnetic field multiplied by a unit charge, it will be in the units of $\mathrm{kg}\cdot\mathrm{s}^{-1}$ because the unit of charge vanishes. Therefore, B is also dimensionally consistent. Then, we have

$$\xi = \frac{2(n_{r;2} - n_{r;1}) + (|m_{l;2}| - |m_{l;1}|)}{m_{l;2} - m_{l;1}} = \frac{\left(\hbar B + m^2\omega^2\theta\right)^2}{\hbar^2 B^2 + 4\hbar^2 m^2\omega^2 - 2\hbar m^2\omega^2 B\theta + m^2\omega^2\theta^2}. \quad (4.8)$$

To produce degenerate energy levels, the right-hand side has to be equal to the left-hand side, which is rational. By substituting the proposed definition of B and θ, the simplified form of ξ is

$$\xi = \sqrt{\frac{(f+g)^2}{4+(f-g)^2}}. \quad (4.9)$$

Now, the problem of finding degenerate states essentially reduces to the problem of finding the appropriate values or sequence of values of f and g such that (4.8) is rational.

By the method of induction, we find that the difference $f-g$ has to be either one in the set below

$$f - g = \left\{\frac{4nk}{n^2-k^2}, \frac{n^2-k^2}{nk}\right\}, \quad (4.10)$$

where n and k are positive integers as well as co-prime with f greater than g but not both odd apart from that, since $0 < B\theta < \hbar$ and $B\theta = fg\hbar$ then $fg < 1$. We have complete control over the magnetic field's value, so we will attach the subscript to the controlling parameter f_{exp}. As a result, the coefficient of repetitive noncommutativity $g_{n,k;f}$ is given by

$$g_{n,k;f} = \left\{f_{\mathrm{exp}} - \frac{4nk}{n^2-k^2}, f_{\mathrm{exp}} - \frac{n^2-k^2}{nk}\right\}. \quad (4.11)$$

We also want to emphasize that in (4.8), f_{exp} and $g_{n,k;f}$ must each be a positive rational number in order to ensure that the numerator is rational and thus ξ is rational. Then, the degenerate energy levels and repetitive noncommutativity will be

$$E_d = (2n_r + |m_l| + 1)\,\hbar\Omega - \sqrt{\frac{(f_{\mathrm{exp}} + g_{n,k;f})^2}{4 + (f_{\mathrm{exp}} - g_{n,k;f})^2}}\, m_l\hbar\Omega, \quad (4.12)$$

$$\theta_d = \left\{\left(\frac{\hbar}{m\omega}\right) g_{n,k;f} \,\middle|\, g_{n,k;f} = \left\{f_{\mathrm{exp}} - \frac{4nk}{n^2-k^2}, f_{\mathrm{exp}} - \frac{n^2-k^2}{nk}\right\}, \gcd(n,k) = 1, n,k \in \mathbb{Z} > 0\right\}. \quad (4.13)$$

Notice that in order to have degenerate states, B and θ have to simultaneously follow (4.7) and the fact that they are positive rational. This immediately implies that satisfying the simultaneous condition for $0 < B\theta < \hbar$ in the case III is in principle possible, but at the expense of tweaking the magnetic field to very specific strengths because the other physical constants in (4.7) cannot be determined with absolute precision to be denoted in rational fraction because there is always some level of uncertainty.

In other words, we can actually infer that for $0 < B\theta < \hbar$ in the case III, in principle, the system is highly non-degenerate as only the finely tuned magnitude of the magnetic field can produce degenerate states. However, in practice, it is sufficient to claim that the system is effectively non-degenerate for any particle of interest and any applied magnetic field in this case. Just like in the case I, there is no singular answer to the exact degeneracy of the system as it not only depends on the two free integral parameters, n and k, but also depends on the applied magnetic field. We can then also conclude that if the energy eigenvalues and noncommutativity parameter possess any arbitrary values which do not match with (4.12) and (4.13) respectively, then we say those values are not associated with any degeneracy.

This is in contrast with the case II where the system is highly, and in fact, infinitely degenerate, be it in principle or in practice. This means that for the purpose of studying degenerate states in the presence of a magnetic field, all degenerate states can be studied in the case II alone in practice. As has been mentioned previously, we can set the magnetic field to be of any arbitrary value and the corresponding repetitive noncommutativity can be found by the relation $B\theta = \hbar$. As a result, we will only use the instructive example in the following section to demonstrate how strict the degeneracy requirement must be and how to apply θ_d in (4.13).

4.3 Numerical example

We will again consider the same parameters as used in the case I for a single GaAs parabolic dot. Recall that the effective mass and frequency are $m_{\text{eff}} = 6.103 \times 10^{-32}\,\text{kg}$ and $\omega_0 = 4.555 \times 10^{12}\,\text{s}^{-1}$ respectively. Using our definition of B, the cyclotron energy in [32, 33] is given by

$$\hbar\omega_c = \frac{\hbar B}{m} = 1.76\,\text{meV} = 2.820 \times 10^{-22}\,\text{J}, \tag{4.14}$$

at $B = 1\,\text{T}$. In this case, B is still a magnetic field multiplied by a unit charge. However, the magnitude of the unit charge is no longer $q = 1.602 \times 10^{-19}\,\text{C}$ but instead $q_{\text{eff}} = 1.631 \times 10^{-19}\,\text{C}$ so that the above equation is correct. Then, the controlling parameter f_{exp} for a unit Tesla is

$$f_{\text{exp}} = \frac{B_{\text{exp}}}{m_{\text{eff}}\omega_0} = 0.587 \approx \frac{1}{2}. \tag{4.15}$$

By taking the above approximation, the magnetic field B_{exp} has to change to $B_{\text{exp}} = 0.852 q_{\text{eff}}\text{kg s}^{-1}$. We will use the second element of the set in (4.10) and by letting $n = 5$ and $k = 4$, then

$$g_{n,k;f} = \frac{1}{2} - \frac{5^2 - 4^2}{5(4)} = \frac{1}{20}. \tag{4.16}$$

We want to emphasize that n and k can also take other integral values such that $g_{n,k:f}$ is still positive, which will then result in different repetitive noncommutativity for the same system, albeit the energy gap and hence the distribution of degenerate energy levels will be different. Note that

$$f_{\text{exp}} - g_{n,k;f} = \frac{9}{20}, \tag{4.17}$$

which does in fact satisfy equation (4.10). Then, we have

$$\xi = \sqrt{\frac{(f_{\text{exp}} + g_{n,k;f})^2}{4 + (f_{\text{exp}} - g_{n,k;f})^2}} = \frac{11}{41}. \tag{4.18}$$

This results in the following degenerate energy levels and repetitive noncommutativity

$$E_d = (2n_r + |m_l| + 1)\,\hbar\Omega - \frac{11}{41} m_l \hbar\Omega, \tag{4.19}$$

$$\theta_d = g_{n,k;f} \frac{\hbar}{m_{\text{eff}}\omega_0} = 1.897 \times 10^{-17}\,\text{m}^2. \tag{4.20}$$

We will not, however, be plotting the energy level diagram for the above value of ξ since its numerator and denominator are large. That means, it is expected that the successive degenerate states have a relatively high energy gap to be plotted on a proper scale. We want to note again that there is a variation in the distribution of degenerate energy levels for different n and k. As a result, we must consider having a charged particle with a sufficiently low mass, frequency, and magnetic field so that (4.15) and thus ξ have more feasible values for the energy gap between successive degenerate states to be smaller. As previously stated, we will not look further into this section because it is only intended to demonstrate the strict requirements of θ_d and how the formula can be used. It is indeed justifiable to claim that the system in the case III is in principle highly non-degenerate and in practice effectively non-degenerate.

5 Effects of B and θ on probability densities

In this section, we are going to display the probability distribution functions of the ground and excited states and also the effects of magnetic field and noncommutativity for each of the cases. Despite the constraints imposed in these three cases, the energy eigenvalues and eigenstates can all be traced back to (2.29) and (2.30), where we only need to manipulate Ω and γ, which only affect the coefficients rather than the variables of the polynomial functions themselves. Hence, it is expected that the upcoming plots of probability densities will have a similar characteristic shape.

We will plot the probability densities of the first 25 ground and excited states to observe their behavior. Regarding the magnetic field and noncommutativity effect, we will only present the ground state, i.e., $|\Psi_{0,0}|^2$, because the effect can naturally be extended to higher-order states. For all the cases, we will set the mass and frequency of the particle to be similar to those in the numerical examples of degenerate energy spectra in the approximate model of a single GaAs parabolic quantum dot from the previous sections, i.e,

$$m_{\text{eff}} = 6.103 \times 10^{-32}\,\text{kg},\ \omega_0 = 4.555 \times 10^{12}\,\text{s}^{-1}. \tag{5.1}$$

The density plots of the probability distribution functions for the ground state and the first few excited states are manifested in figure 4, figure 6 and figure 8 in cases I, II and III, respectively. Accordingly, the effects of noncommutativity on the ground state probability density functions are shown in figure 5, figure 7 and figure 9. We will briefly discuss the behavior of the probability density function in each of the figures of the plots from the previous section. Recall that n_r is known as the radial quantum number, and it corresponds to the number of nodes and antinodes of the wave function moving radially outward from the center. A positive or negative m_l can be viewed as the counterclockwise or clockwise motion of the wavefunction. For the probability density function, we can only observe the number of nodes designated as a few concentric rings due to its rotational symmetry arising from the usage of polar coordinates. Hence, when $n_r$increases, so does the number of nodes. It is also clear that as $|m_l|$ increases, so does the radius of the concentric ring. This implies that this is true regardless of the sign of m_l.

When we manipulate B or θ, however, we only prepare density plots of the probability distribution functions at the ground state $|\Psi_{0,0}|^2$ because its influence naturally extends to higher-order states. The local maximum of the function is always concentrated at its center in each figure at $|\Psi_{0,0}|^2$, i.e., the origin designated by the brightest spot on the plot, implying that the particle is most likely to be found there. In the case I, as θ increases, the probability amplitude of this Gaussian-like function spreads out farther radially outward, which suggests that the likelihood of finding the particle further from the origin increases. This behavior also persists for higher-order states.

In the cases II and III, we vary the strength of the magnetic field to observe how it affects the probability densities. In figure 7, we realize that as B increases, the probability amplitude becomes more concentrated at its center. We can indirectly observe the effect of noncommutativity by simply substituting B for θ in the aforementioned condition, which clearly has the opposite effect as B. Such opposite behavior is naturally present in the case III as well, and as a result, it is unnecessary to construct the plots.

5.1 Case I: $B\theta = 0$

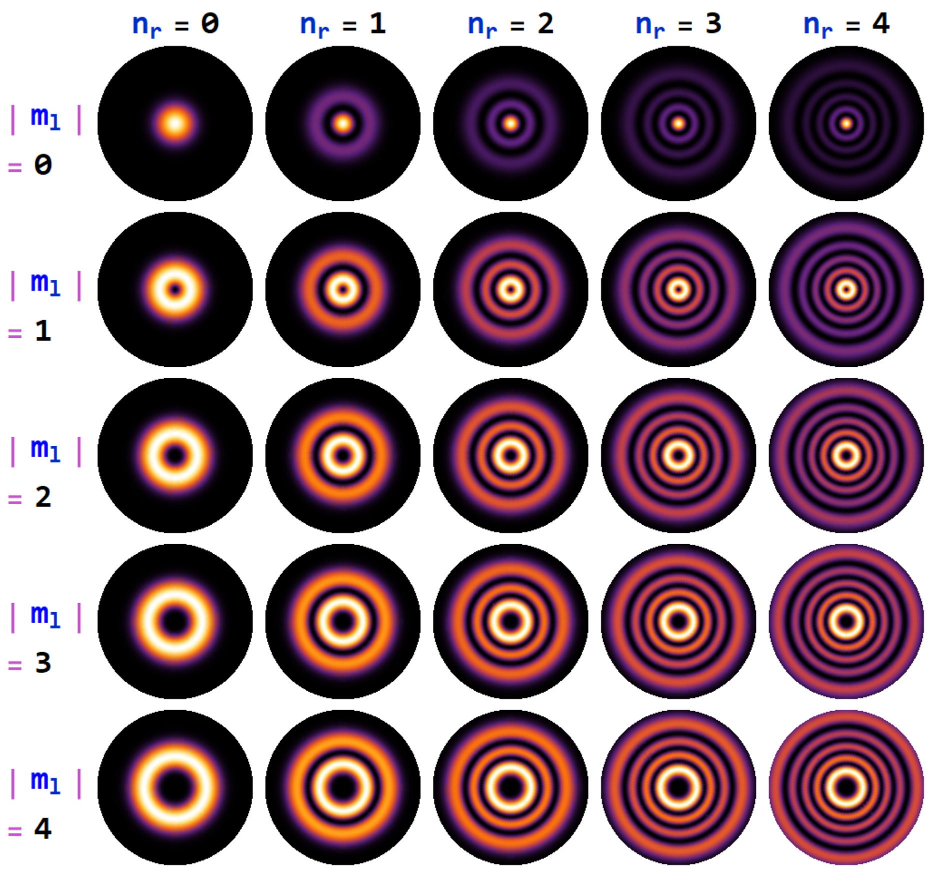

Figure 4: (color online) Density plot of $|\Psi_{n_r,|m_l|}|^2$ at $B\theta = 0$ where $\theta = 2.683 \times 10^{-16}\,\mathrm{m}^2$ bounded by the domain with radius 100 nm.

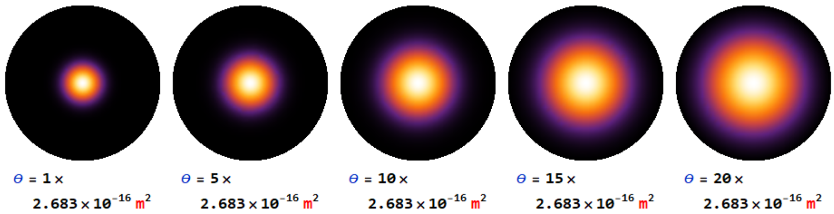

Figure 5: (color online) Effect of θ on $|\Psi_{0,0}|^2$ at $B\theta = 0$ bounded by the domain with radius $r = 100\,\mathrm{nm}$.

5.2 Case II: $B\theta = \hbar$

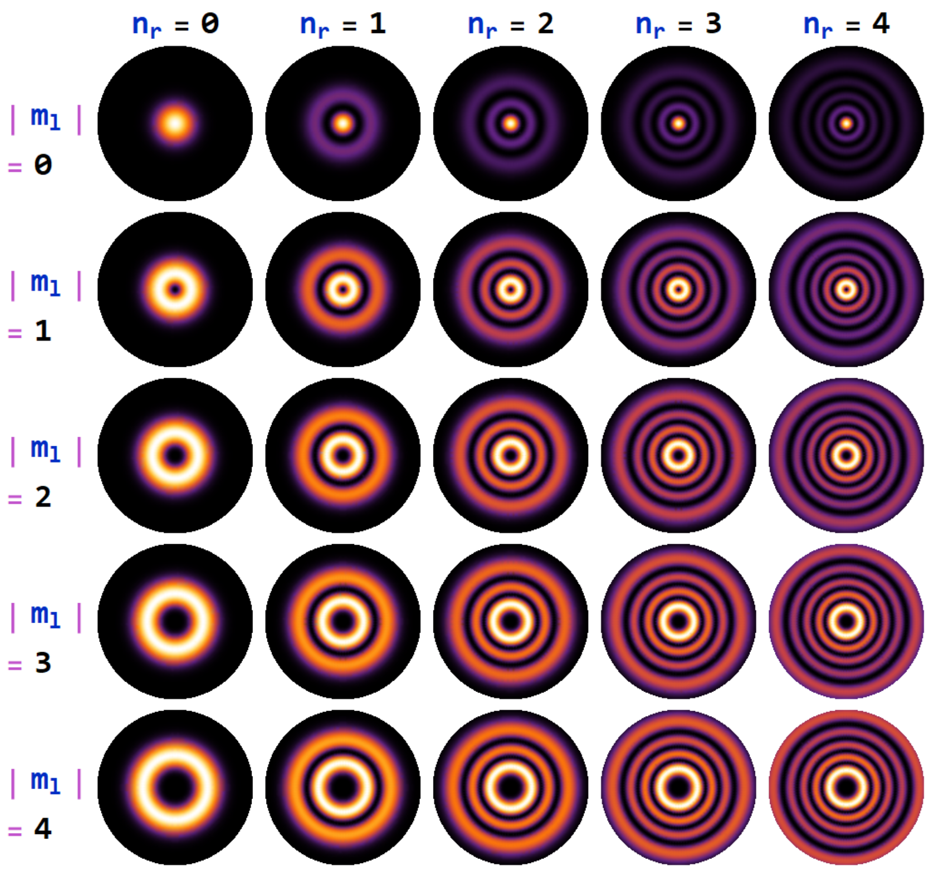

Figure 6: (color online) Density plot of $|\Psi_{n_r,|m_l|}|^2$ at $B\theta = \hbar$ where $B = 12q_{\mathrm{eff}}\,\mathrm{T}$ bounded by the domain with radius $r = 25\,\mathrm{nm}$.

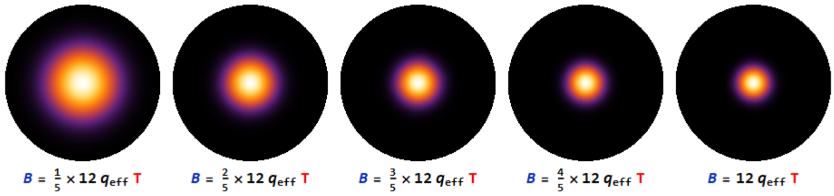

Figure 7: (color online) Effect of B on $|\Psi_{0,0}|^2$ at $B\theta = \hbar$ bounded by the domain with radius $r = 25\,\text{nm}$.

5.3 Case III: $0 < B\theta < \hbar$

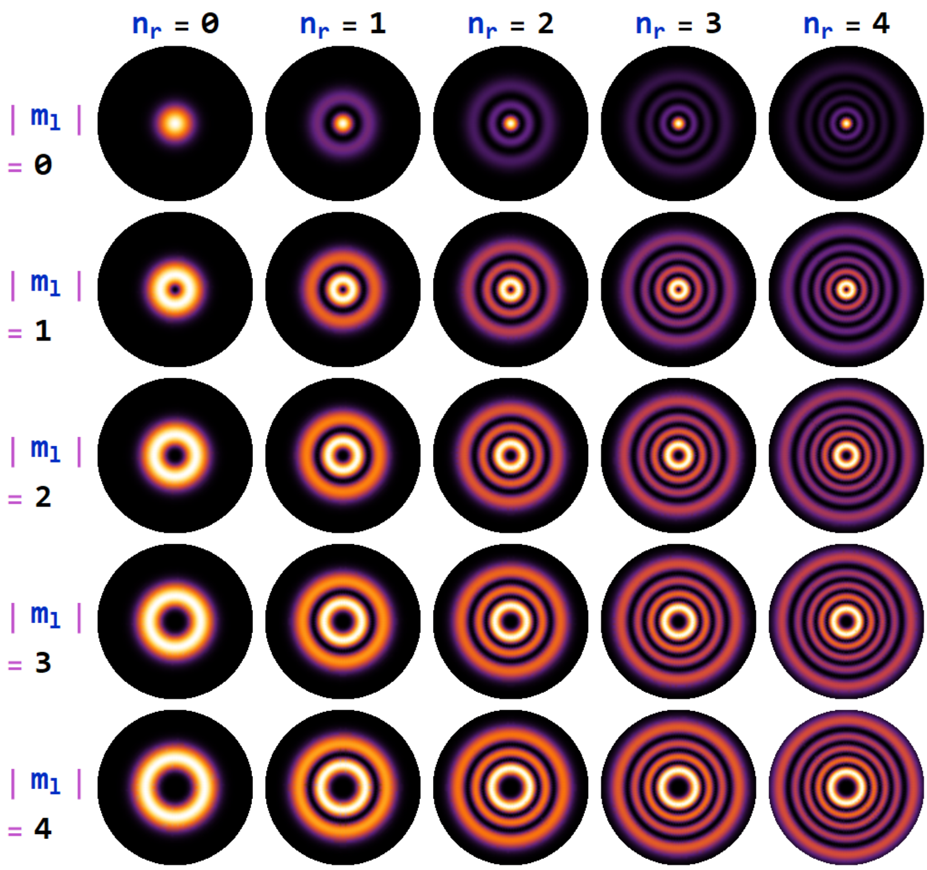

Figure 8: (color online) Density plot of $|\Psi_{n_r,|m_l|}|^2$ at $0 < B\theta < \hbar$ where $B = q_{\text{eff}}\,\text{T}$ bounded by the domain with radius $r = 100\,\text{nm}$.

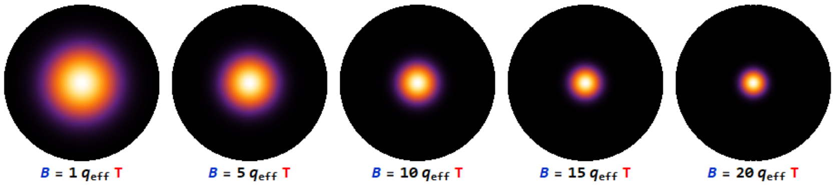

Figure 9: (color online) Effect of B on $|\Psi_{0,0}|^2$ at $0 < B\theta < \hbar$ bounded by the domain with radius $r = 50\,\mathrm{nm}$.

6 Conclusion

The energy eigenvalues and quantum states of the Landau problem in 2D noncommutative space were demonstrated to be unique features of the system using a 2-parameter family of unitarily equivalent irreducible representations of the nilpotent Lie group G_{NC}. They depend on the particle's mass m and frequency ω as well as the applied magnetic field. The polynomial structure of the rotationally symmetric energy eigenstates is not affected by the two physical parameters of the particle, as they only affect the coefficients of the variables. Because the oscillator's effective mass M must be real, we must impose the following condition: $0 \leq B\theta \leq \hbar$. At $B\theta = 0$, which implies that $B = 0$, the solution to the eigenvalue problem is simply the noncommutative planar harmonic oscillator based on minimal coupling prescription. We can have different answers to the degeneracy of the system as it depends on the two free parameters n and k (where each of them is any arbitrary positive integer) and is irrespective of the choice of particle. When it comes to $B\theta = \hbar$, the system is infinitely degenerate for any arbitrary value of B. The remaining case, however, is in principle highly non-degenerate and in practice effectively non-degenerate. It depends on two parameters, n and k (where they are co-prime and not simultaneously odd integers) and the applied magnetic field. It comes with the limitation that the strength of the magnetic field has to be finely-tuned.

For future prospects, further research can be implemented to construct a more generalized gauge invariant transformation. The study of gauge invariant degeneracies and symmetric wavefunctions can be extended to different variants of the problem. For example, other exactly solvable eigenvalue problems with different potentials, parameter-dependent eigenvalue problems (e.g., energy-dependent harmonic oscillator, time dependence in mass and frequency, etc.), relativistic models, time evolution, etc.

We have numerically studied how the probability densities and degeneracies are manifested in the approximate (confining potential) model of a single GaAs parabolic quantum dot but in gauge invariant 2D noncommutative phase space. For cross-disciplinary study, it would be of phenomenological and pedagogical interest to investigate the effects of noncommutativity in other mesoscopic systems, such as the few-electron quantum dot, while taking into account other factors such as Zeeman effect, spin-orbit coupling, Coulomb interaction, and so on to explore how it behaves in this elusive space.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

Authors declare that there is no conflict of interest.

Acknowledgments

NMS wish to acknowledge the Institute for Mathematical Research (INSPEM), Universiti Putra Malaysia (UPM) through Interim Researcher Funding Initiative (INSPEM/IRFI/2/2020/6233205) for part of the work.

References

[1] H. S. Snyder, "Quantized space-time", Physical Review **71**, 38 (1947).

[2] R. Szabo, "Quantum field theory on noncommutative spaces", Physics Reports **378**, 207 (2003).

[3] L. Gouba, "A comparative review of four formulations of noncommutative quantum mechanics", International Journal of Modern Physics A **31**, 1630025 (2016).

[4] E. Akofor, A. P. Balachandran and A. Joseph, "Quantum fields on the Groenewold-Moyal plane", International Journal of Modern Physics A **23**, 1637 (2008).

[5] F. Delduc, Q. Duret, F. Gieres and M. Lefrancois, "Magnetic fields in noncommutative quantum mechanics", Journal of Physics: Conference Series **103**, 012020 (2008).

[6] M. Douglas and N. Nekrasov, "Noncommutative field theory", Reviews of Modern Physics **73**, 977 (2001).

[7] B. S. Lin and T. H. Heng, "Energy spectra of the harmonic oscillator in a generalized noncommutative phase space of arbitrary dimension", Chinese Physics Letters **28**, 070303 (2011).

[8] A. Jahan, "Noncommutative harmonic oscillator at finite temperature: a path integral approach", Brazilian Journal of Physics **38**, 144 (2008).

[9] J. Jing, S. H. Zhao, J.-F. Chen and Z. W. Long, "On the spectra of noncommutative 2D harmonic oscillator", The European Physical Journal C **54**, 685 (2008).

[10] M. N. N. M. Rusli, N. M. Shah, H. Zainuddin, and C. K. Tim, "Analytical spectrum isomorphism of noncommutative harmonic oscillator and charged particle in magnetic field", Jurnal Fizik Malaysia **43**(1), 10043 (2022).

[11] Ö. F. Dayi and A. Jellal, "Landau diamagnetism in noncommutative space and the nonextensive thermodynamics of Tsallis", Physics Letters A **287**, 349 (2001).

[12] Ö. F. Dayi and A. Jellal, "Hall effect in noncommutative coordinates", Journal of Mathematical Physics **43**, 4592 (2002).

[13] A. Jellal and H. B. Geyer, "Second virial coefficient for noncommutative space", Modern Physics Letters A **18**, 927 (2003).

[14] A. Jellal, "Noncommutativity parameter and composite fermions", Modern Physics Letters A **18**, 1473 (2003).

[15] G. F. Wei, C. Y. Long, Z. W. Long and S. Qin, "Exact solution to two-dimensional isotropic charged harmonic oscillator in uniform magnetic field in non-commutative phase space", Chinese Physics C **32**, 247 (2008).

[16] M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, "Hydrogen atom spectrum and the Lamb shift in noncommutative QED", Physical Review Letters **86**, 2716 (2001).

[17] J. Gamboa, M. Loewe and J. C. Rojas, "Noncommutative quantum mechanics", Physical Review D **64**, 067901 (2001).

[18] J. Gamboa, F. Mendez, M. Loewe, and J. C. Rojas, "The Landau problem and noncommutative quantum mechanics", Modern Physics Letters A **16**(32), 2075 (2001).

[19] S. Dulat and L. I. Kang, "Landau problem in noncommutative quantum mechanics", Chinese Physics C **32**, 92 (2008).

[20] P. D. Alvarez, J. Gomis, K. Kamimura, and M. S. Plyushchay, "Anisotropic harmonic oscillator, non-commutative Landau problem and exotic NewtonHooke symmetry", Physics Letters B, **659**(5), 906 (2008).

[21] B. Mirza and M. Mohadesi, "The Klein-Gordon and the Dirac oscillators in a noncommutative space", Communications in Theoretical Physics **42**, 664 (2004).

[22] W. A. N. G. Jian-Hua, L. I. Kang and D. Sayipjamal, "Klein-Gordon oscillators in noncommutative phase space", Chinese physics C **32**, 803 (2008).

[23] H. Hassanabadi, S. S. Hosseini, and S. Zarrinkamar, "Dirac oscillator in noncommutative space", Chinese physics C **32**, 803 (2008).

[24] B. Mirza, R. Narimani, M. Zarei, "AharonovCasher effect for spin-1 particles in a non-commutative space", The European Physical Journal C **48**(2), 641 (2006).

[25] A. Jellal, "Orbital magnetism of a two-dimensional noncommutative confined system", Journal of Physics A: Mathematical and General **34**, 10159 (2001).

[26] A. Kijanka and P. Kosiński, "Noncommutative isotropic harmonic oscillator", Physical Review D **70**, 127702 (2004).

[27] S. Dulat and K. Li, "Quantum Hall effect in noncommutative quantum mechanics", The European Physical Journal C **60**(1), 163 (2009).

[28] S. H. H. Chowdhury, T. A. Chowdhury and M. A. U. Duha, "Gauge invariant energy spectra in 2-dimensional noncommutative quantum mechanics", Annals of Physics **430**, 168505 (2021).

[29] S. H. H. Chowdhury and S. T. Ali, "Wigner functions for noncommutative quantum mechanics: a group representation based construction", Journal of Mathematical Physics **56**, 122102 (2015).

[30] M. Janssen, O. Viehweger, U. Fastenrath, and J. Hajdu, "Introduction to the theory of the integer quantum Hall effect", VCH, Weinheim (1994).

[31] E. Drigho-Filho, S. Kuru, J. Negro, and L. M. Nieto, "Superintegrability of the Fock-Darwin system", Annals of Physics **383**, 101 (2017).

[32] M. Governale, and C. Ungarelli, "Gauge-invariant grid discretization of the Schrödinger equation", Physical Review B **58**(12), 7816 (1998).

[33] L. P. Kouwenhoven, D. G. Austing, and S. Tarucha, "Few-electron quantum dots", Reports on Progress in Physics **64**(6), 701 (2001).

[34] J. Z. Zhang, "Fractional angular momentum in non-commutative spaces", Physics Letters B **584**(1-2), 204 (2004).

[35] B. K. Pal, B. Roy, and B. Basu, "Quantum dot with spinorbit interaction in noncommutative phase space and analog Landau levels", Physics Letters A **374**(42), 4369 (2010).

[36] S. D. Liang, H. Li, and G. Y. Huang, "Detecting noncommutative phase space by the Aharonov-Bohm effect", Physical Review A **90**(1), 010102 (2014).

[37] S. Khan, B. Chakraborty, and F. G. Scholtz, "Role of twisted statistics in the noncommutative degenerate electron gas", Physical Review D **78**(2), 025024 (2008).

[38] P. Basu, B. Chakraborty, and S. Vaidya, "Fate of the superconducting ground state on the Moyal plane", Physics Letters B **690**(4), 431 (2010).

[39] Y. G. Miao, and H. Wang, "Energy spectrum and phase transition of superfluid Fermi gas of atoms on noncommutative space", Symmetry, Integrability and Geometry: Methods and Applications **10**, 075 (2014).

[40] V. Fock, "Bemerkung zur quantelung des harmonischen oszillators im magnetfeld", Zeitschrift für Physik **47**, 446 (1928).

[41] C. G. Darwin, "The diamagnetism of the free electron", Mathematical Proceedings of the Cambridge Philosophical Society **27**, 86 (1931).

[42] A. Benchikha and M. Merad, "Energy-dependent harmonic oscillator in noncommutative space: a path integral approach", International Journal of Modern Physics A **32**, 1750194 (2017).

[43] A. Bhuiyan and F. Marsiglio, "Landau levels, edge states, and gauge choice in 2D quantum dots", American Journal of Physics **88**, 986 (2020).