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Letter

How are the degrees of freedom responsible for entropy in BTZ spacetime?

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Abstract. The entanglement entropy approach to study the dependence of entropy upon the location of degrees of freedom (dof) (near/far) from the horizon is discussed in this article. We try to understand the physical deviation of the area law for the excited states by incorporating the logarithmic and power law corrections. We show that the dof near the horizon give contribution to the total entropy of the system in the ground state, and away from the event horizon gives contribution to the excited state.

Keywords: BTZ Black Hole; Entanglement Entropy; Degrees of Freedom.

Black holes are the non-singular solutions to Einstein's field equation, and they behave as the thermodynamic objects. It means we can easily study black hole mechanics by applying the laws of thermodynamics proposed by Bekenstein [1, 2, 3, 4] and Hawking [5]. The temperature and entropy of the black holes are identified as the surface gravity and area of the horizon [1, 5]. It radiates when the quantum mechanical effect is taken into account, and this radiation is known as Hawking radiation. Many attempts have been made to understand this radiation by studying the entropy of the black hole.

Out of several methods used to explore the entropy in BTZ space time [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], we focus our study on analyzing the same using the entanglement entropy approach, which is the most attractive candidate among these methods. The EE is a non-geometric method, and in this approach, we measure the quantum information due to the division of a system [18, 19, 20, 21, 22]. The EE of other gravity theories is explored in Ref. [23, 24, 25].

The EE is one of the promising candidates to study the source of black hole entropy and its subleading corrections (logarithmic and power law corrections) [26, 27, 28, 29, 30, 31, 32, 33]. It arises due to the vacuum and thermal fluctuations [34, 35, 36, 37, 38, 39, 40, 41] in the vicinity of black hole space time. We have made our attempts to study the scalar fields propagating in the background of BTZ space time. We study the location of dof near and far from the event horizon of BTZ space-time. In this attempt to investigate the dependence of entropy on dof, two approaches have been developed. The first approach is associated with fundamental dof related to string loops [42, 43, 44] and the other one associated with quantum fields (scalar) propagating in the black hole space-time. We are using the second approach. One counts certain dof on the event horizon but can not say the precise location relevant to the dof. The power law corrections arise due to thermal fluctuation [34] in the presence of excited state (ES) [45, 46] and the logarithmic corrections arise due to the quantum fluctuations [47, 48].

In this paper, we have made our attempts to study the problems as mentioned above, in a more general framework, which may not be relevant to other approaches. We start our study by considering the entanglement between quantum fields lying inside and outside the black hole. The violation of area law [45, 46] for ES can be understood by ascertaining the location of dof far and near the event horizon of the black hole, which leads to EE in these cases.

The metric of the BTZ space time is given by the following line element [49]

$$ds^2 = -\left(-M + \frac{r^2}{l^2}\right) dt^2 + \left(-M + \frac{r^2}{l^2}\right) dr^2 + r^2\, d\phi^2. \tag{1}$$

The metric of the BTZ black hole can be written in terms of proper length

$$ds^2 = -k^2\, dt^2 + \, d\rho^2 + l^2(k^2 + M) d\phi^2, \tag{2}$$

where $r^2(\rho) = l^2(k^2 + M)$ and M is the mass of the BTZ black hole. The scalar filed in the background of the BTZ black hole is

$$S = -\frac{1}{2}\int dt\, \sqrt{-g}\, \left(g^{\mu\nu}\left(\partial_\mu \Phi\, \partial_\nu \Phi\right) - \mu^2 \Phi^2\right), \tag{3}$$

using the separation of variables, the field Φ decomposed as

$$\Phi(t, \rho, \phi) = \sum_m \phi_m(t, \rho)\, e^{im\phi}, \tag{4}$$

and this decomposition of Φ manifests the cylindrical symmetry of the system. The scalar field in the presence of BTZ space time is

$$S = -\frac{1}{2}\int dt \left[\frac{\sqrt{(k^2 + M)}}{k}\, \dot{\Phi}_m^2 + k\sqrt{k^2 + M}(\partial_\rho \Phi_m^2) + \frac{k^2 m^2}{k\sqrt{k^2 + M}}\, \Phi_m^2\right], \tag{5}$$

and the corresponding Hamiltonian is

$$H = \frac{1}{2}\int d\rho\, \tilde{\pi}_m^2(\rho) + \frac{1}{2}\int d\rho\, k\, \sqrt{k^2 + M}\left(\partial_\rho\left(\frac{k}{\sqrt{k^2 + M}}\right) \psi_m\right)^2 + \frac{m^2 k^2}{M + k^2}\psi_m^2, \tag{6}$$

where

$$\psi_m(t, \rho) = \left(\frac{k^2}{k^2 + M}\right)^{1/4} \Phi_m(t, \rho), \tag{7}$$

where $\tilde{\pi}_m$ is canonical momentum corresponding to the field and it satisfy the following relation $[\phi_m(\rho), \tilde{\pi}_{m'}(\rho')] = \delta_{m,m'}\delta(\rho - \rho')$. The system can be discretized by the following replacement

$$\rho \to (A - \frac{1}{2})a, \qquad \delta(\rho - \rho') \to \frac{\delta_{AB},}{a} \tag{8}$$

where $A, B = 1, 2....N$ and "a" is ultra-violet cut-off. The replacements of the field are

$$\psi_m(\rho) \to q^A, \qquad \tilde{\pi}_m(\rho) \to \frac{p_A}{a}, \qquad V(\rho, \rho') \to \frac{V_{AB}}{a^2}. \tag{9}$$

These replacements lead to the discretized Hamiltonian, which is identical to the N coupled Harmonic oscillator. It is written as,

$$H = \sum_{A,B=1}^{N} \left[\frac{1}{2a} \delta^{AB} p_A\, p_B + \frac{1}{2} V_{AB}\, q_m^A\, q_m^B \right], \tag{10}$$

where $p_A = a\delta_{AB}\dot{q}^B$ is the canonical momentum corresponding to q^A. The interaction matrix elements V_{AB} (where $A, B = 1, 2 \ldots\ldots N$) are obtained by comparing the Eq. (3) and (8) which are written as [47],

$$\begin{aligned} V_{AB} = \quad & \frac{k_A}{\sqrt{k_A^2 + M}} \left(k_{A+1/2} \sqrt{k_{A+1/2}^2 + M} + k_{A-1/2} \sqrt{k_{A-1/2}^2 + M} \right) \delta_{A,B}, \\ & -2k_{A+1/2} \sqrt{k_{A+1/2}^2 + M} \sqrt{\frac{k_A}{\sqrt{k_A^2 + M}}} \sqrt{\frac{k_{A+1}}{\sqrt{k_{A+1}^2 + M}}} \delta_{A,B+1} \\ & +m^2 \frac{k_A^2}{k_A^2 + M} \psi_m^{A^2} \delta_{A,B}. \end{aligned} \tag{11}$$

The diagonal and off diagonal terms of the matrix V_{AB} can be identified from the equation (3). The various matrix elements of V_{AB} matrix are written as,

$$\begin{aligned} \Sigma_A^{(m)} &= \frac{k_A}{\sqrt{(u_A^2 + M)}} \left(k_{A+\frac{1}{2}} \sqrt{k_{A+1/2}^2 + M} - k_{A-1/2} \sqrt{[(k_{A-1/2}^2 + M)} \right) + m^2 \frac{k_A^2}{(k_A^2 + M)}, \\ \Delta_A &= -\, k_{A+1/2} \sqrt{[(k_{A+1/2}^2 + M)} \sqrt{\frac{k_{A+1}}{\sqrt{(k_{A+1}^2 + M)}}} \sqrt{\frac{k_A}{\sqrt{(k_A^2 + M)}}}. \end{aligned} \tag{12}$$

These off-diagonal terms of the matrix leads to interaction between the states in the same way as the nearest neighboring of the harmonic oscillator interacts.

In order to understand how the dof near/far from the event horizon contribute to the correction terms and how the correction arises/vanishes in both cases, we have to consider the state of quantum fields around the black hole. The interaction matrix 11 tells us about the location of degrees of freedom. The matrix form of the V_{AB} is,

$$V_{(AB)}^m = \begin{pmatrix} \Sigma_1 & & & & & & \\ & \Sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & |\overline{\Delta A - 2} & \overline{\Sigma_{A-2}} & \quad | & \\ & & & |\Delta_{A-1} & \Sigma_{A-1} & \Delta_A \; | & \\ & & & | & \underline{\Sigma_A} & \underline{\Delta_{A+1}}| & \\ & & & & & & \ddots \end{pmatrix}. \tag{13}$$

where the matrix element in the bracket is called the window. The percentage contribution of the entropy as a function of q by using the following relation,

$$pc(q) = \frac{S(q, fixed\ d)}{S_{total}} \times 100\,, \tag{14}$$

where S_{total} is the total entropy of the system where $i, j = 0..........N$. The entropy of the GS is given by [46]

$$S = \sum_{i=1}^{N-n_B} S_i \quad \text{with} \quad S_i = -\frac{\nu_i}{1-\nu_i} ln\, \nu_i - ln\,(1-\nu_i), \tag{15}$$

where

$$\nu_i = \frac{1}{\lambda_i}(\sqrt{1+\lambda_i} - 1)^2, \qquad\qquad 0 < \nu_i < 1. \tag{16}$$

The Eq. (14) shows the percentage contribution of the total entropy, which is the function of window position (q). In our calculations we have taken fix value of N and n, i.e $N = 300$ and $n = 100,\ 150$. Studying the Eq. (14), we can say that,

- The percentage contribution of interaction term to the entropy is zero (from von Neumann entropy relation). The presence of interaction terms raises the entropy significantly. We also observe that the inclusion of *dof* inside and outside the event horizon contributes to entropy.
- The percentage contribution of entropy depends upon the window position. In the absence of interaction, the peak is placed symmetrically inside and outside the horizon. This observation suggests that the maximum contribution to the entropy comes from those *dof* near the event horizon.
- For the excited state, the peak lowers as we increase the excitation number o. This suggests that, increasing the excitation number (o), the significant contribution comes from the *dof* far away from the event horizon.

From the above discussion, we confirm that the entanglement between *dof* inside and outside the event horizon, the contributes to the entropy. The contribution to the entropy is more from the *dof* near the event horizon, and they decrease with the increasing excitation number.

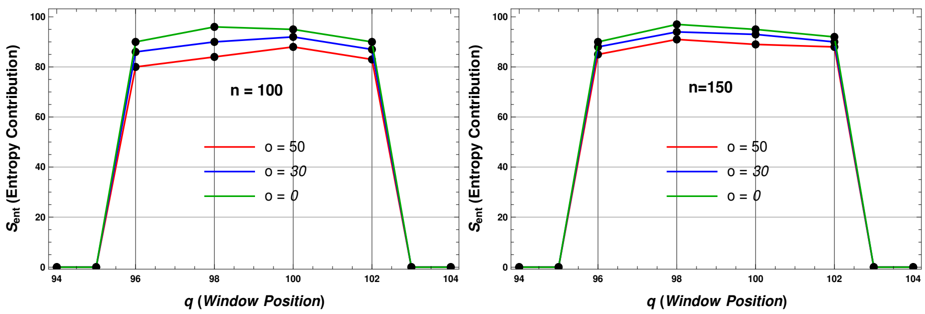

Figure 1: Plot of total entropy as a function of window position for fixed window size $N = 300$, $n = 100$ and 150, for the ground state and excited state with excitation number $o = 30$ and 50. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article [https://jhap.du.ac.ir/].)

Now, we study how *dof* contribute to the entropy as a function of window width. The percentage contribution to the EE for every window width d can be calculated from the relation,

$$pc(d) = \frac{S(d)}{S_{total}} \times 100. \tag{17}$$

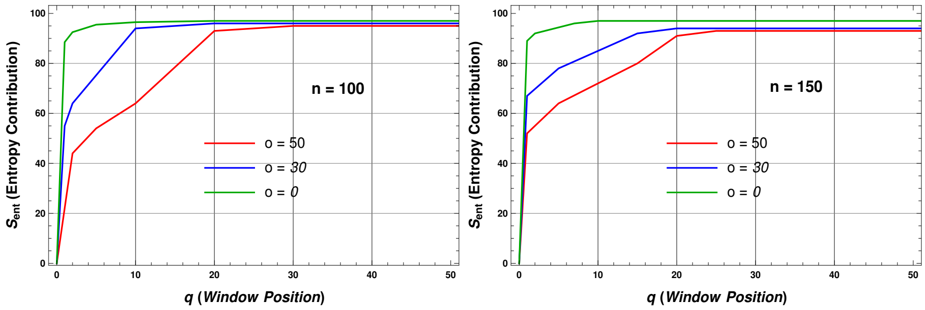

Figure 2: Plot of the percentage contribution of $S_{ent}(t)$ for the GS and ES, $N = 300$, $n = 100$ and 150 with excitation number $o = 30$ and 50. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article [https://jhap.du.ac.ir/].)

The effect of dof on the entropy of the ES is shown in Fig. 1. We plot the graph of the percentage contribution of entropy with the window width in d and Fig. 2 for the GS, ES with $n = 100$ and 150 for fixed N. This graph shows that the GS follows the area law at the small value of the window width $d = 5$, but for the ES, it is recovered at the higher value of $d = 20$ for the excitation number $o = 30$ and $d = 25$ for the excitation number $o = 50$. From this, we can conclude that the contribution of the dof near the event horizon is more for the ES and increases with the excitation number (o). The entropy depends upon the location of dof for ES. The logarithmic correction does not significantly affect the ES, but is present in GS [47, 50]. The logarithmic corrections are present only in the case of small black holes (the ES also contributes to the entropy in the form of power law corrections), but for the large limit, area law holds (see Fig. 1 and Fig. 2). The logarithmic corrections in the entropy arise due to the high energy quantum fluctuations of fields near the horizon. These quantum fluctuations are small for macroscopic black holes, and the leading term, which describes the situation of thermal fluctuations is averaged out.

We studied the location of dof near/far from the horizon. We have shown how the dof are responsible for the entropy. The behavior of the dof is depicted in Fig. 1 and Fig. 2. We can see clearly from Fig. 1 that the peak is symmetric inside and outside the horizon, for the ground state. But the ES, the peak diminishes with increasing excitation number $o = 30, 50$. The peak increases when the window is near the event horizon and decreases when far from the event horizon. From this observation, we can say clearly the dof are responsible for entropy and contribute more for the GS rather than ES. Fig. 2 shows that the GS entropy and excited state entropy plots have similar behavior at large window width (position of dof), but the effect is clearly visible at small window width. This is related to power law correction, which arises due to the thermal fluctuations applicable near the event horizon, where a large number of the field modes (dof) are present. These results would be more significant in the case of higher dimensional black holes. One can also study the dependence of entropy on dof for fermions and gauge fields in BTZ space-time and non-singular black holes (where the entropy does not follow the area law) [51, 52, 53, 55, 54, 56].

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