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Quasinormal modes of Dirac particle near Reissner-Nordström black hole

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Abstract. In this paper, the behavior of the Dirac particles in the Reissner-Nordström (RN) background is evaluated. Moreover, the thermal properties such as energy, heat capacity, and entropy are investigated. In this regard, the assessment of the holography and information theory near the event horizon, affirms that each particle in a near RN black hole has 4 bits of information.

Keywords: Reissner-Nordström Black Hole; Dirac Equation; Heat Capacity; Entropy; Energy Spectrum.

1 Introduction

One of the greatest accomplishments in the general relativity is the physical prediction of the black hole in the universe. In this regard, theoretical physics has been studying the black hole for decades. Furthermore, the black holes behave as quantum phenomena due to Hawking radiation [1, 2]. Therefore, the investigation of the black holes enables us to figure out the relationship between general relativity theory and quantum mechanics. The consideration of the quantum fields theory in the curved space-time also the behavior of different fields interacting with the gravitational field oblige physicist to assemble variations of theories that incorporates both quantum mechanics and general relativity [3, 4]. That's why black holes are being studied to obtain information, and their implications are mentioned in these articles [5, 6, 7].
The Black holes have three observable quantities, mass, electric charge and angular momentum, which divided them into different categories. The black hole with a charge and mass is called a Reissner-Nordström black hole, similar to the Schwarzschild black hole but with two outer and inner event horizons. So, in the limit $(Q \to 0)$ we will have the same Schwarzschild black hole also in the limit of $(Q \to M)$ will have extreme Reissner-Nordström black hole. When we solve a scalar or Dirac field in the curved space-time, we get to imaginary energy, which indicates quasinormal mode, and was used by Simone, Will, and Fiziev [8, 9], which is a fascinating property of black holes [10, 11, 12, 13, 14]. The quasinormal state is a set of discrete frequencies with two real and imaginary parts, in which the real part corresponds to the actual frequency and the imaginary term mentions the attenuation rate when $\omega_I > 0$ and indicates the instability situation when $\omega_I < 0$ [16]. Furthermore, the quasinormal modes

provide a unique black hole fingerprint for recognition. Hopefully, in the near future, large-scaled interferometric detectors will be exploited to detect gravitational waves. In order to extract as much information as possible from the gravitational wave signals, it is important to figure out the proportionality of quasinormal modes with the black hole parameters.

In recent years, the thermodynamics of black holes has been repeatedly applied to test the predictions of quantum gravity candidate theories [17, 18, 14].
In order to provide a narrative visual perspective, the holographic principle assisted us in dealing with problems regarding to the quantum gravity without quantizing the gravity itself directly [19, 20]. Since understanding the black hole information is a significant process, therefore, finding a holographic interpretation of black holes can be a convenient and promising approach [21].

In [22], we have considered the Klein-Gordon particles near the RN black hole and obtained regarded entropy, and showed that each particle contains 8 bits of information stored on the black hole. Now in this paper, we investigate the Dirac particles near the RN black hole and define the entropy and thermal properties of a black hole in RN space-time.
This paper is organized as follows: In sections 2 and 3, we review the RN black hole properties and the Dirac equation in curved space-time, respectively. In section 4, the energy spectrum of the Dirac particles near the RN black hole is determined by applying the Dirac equation in curved space-time and computing the solutions for radial parts. We address the thermodynamical properties of Dirac particles on RN black hole in section 5, and dedicate section 6 to the holography and information theory and evaluating the particles in this frame. Finally, we come to a conclusion in section 7.

2 Reissner-Nordström black hole

The Reissner-Nordström line element expresses an Einstein's field solution which describes the space-time nearby a spherically symmetric with mass M and an electric charge Q, in which both parameters have length dimension. Furthermore, it is assumed that the electromagnetic field is the only matter in the space, which means when $Q \to 0$, the Schwarzschild metric recovered. Moreover, since the Reissner-Nordström space-time is asymptotically flat, it is expected, when the distance from the body reaches infinity, the line element will approach the Minkowski.
The Reissner-Nordström metric is [23],

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2, \tag{1}$$

with metric function,

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{2}$$

which contains two physical and Cauchy horizons in terms of $r_+ = M + \sqrt{M^2 - Q^2}$ and $r_- = M - \sqrt{M^2 - Q^2}$ respectively,

$$f(r) = \frac{(r - r_-)(r - r_+)}{r^2}. \tag{3}$$

The metric tensor in this background is written as follows:

$$g_{\mu\nu} = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 sin^2\theta \end{pmatrix}. \tag{4}$$

Moreover, the corresponding Hawking temperature and Bekenstein-Hawking entropy of RN black hole as a thermodynamical system represent as [24]

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2}, \qquad S_{BH} = \frac{\pi r_+^2}{G_4}. \tag{5}$$

3 Dirac equation in curved space-time

The wave function Ψ of particles with spin-$\frac{1}{2}$ and rest mass m is described by the Dirac equation as,

$$(i\gamma^\mu D_\mu - m)\Psi = 0, \tag{6}$$

where Dirac matrices $\gamma^\mu = e^\mu_a \gamma^a$ should satisfy anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1_4}$, which $\mathbf{1_4}$ represents the identity operator, also the tetrad parameters e^μ_a can be determined as

$$e^\mu_a = \frac{\partial x^\mu}{\partial y^a}, \tag{7}$$

and it's inverse

$$e^a_\mu = (e^\mu_a)^{-1} = \frac{\partial y^a}{\partial x^\mu}. \tag{8}$$

It is feasible to diagonalize the metric tensor as

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}, \qquad \eta_{ab} = e^\mu_a e^\nu_b g_{\mu\nu}, \tag{9}$$

where $\eta_{ab} = diag(1, -1, -1, -1)$ indicates the local coordinate metric tensor. Moreover, in the presence of charge q and electromagnetic potential A_μ, D_μ determine as,

$$D_\mu = \partial_\mu - \Omega_\mu - iqA_\mu, \tag{10}$$

which ∂_μ indicates the covariant derivative on flat space-time and spin connection Ω_μ defined by

$$\Omega_\mu = \omega_{ab\mu}(x)[\gamma^a, \gamma^b]1/8, \tag{11}$$

here γ^a indicates the standard Dirac matrix (4×4) in flat space-time and represents in terms of Pauli $\sigma_i, i = 1, 2, 3$ and unit matrices $I_{2\times2}$ as [25],

$$\gamma_0 = \begin{pmatrix} I_{2\times2} & 0 \\ 0 & -I_{2\times2} \end{pmatrix}, \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & I_{2\times2} \\ I_{2\times2} & 0 \end{pmatrix}, \tag{12}$$

where have the following relations,

$$\gamma_{0i} = [\gamma_0, \gamma_i] = 2\gamma_0\gamma_i, \qquad \gamma_{ij} = [\gamma_i, \gamma_j] = -2i\epsilon_{ijk}\gamma_5\gamma_0\gamma_k. \tag{13}$$

In Eq. (11), $\omega_{ab\mu}$ is an anti-symmetric parameter in its first two indices, and expressed as follows,

$$\omega_{ab\mu} = \eta_{ac}\omega^c_{b\nu} = \eta_{ac}\left[e^c_\mu \partial_\nu (e^\mu_b) + +e^c_\mu e^\sigma_b \Gamma^\mu_{\sigma\nu}\right], \tag{14}$$

where $\Gamma^\nu_{\sigma\mu}$ is Christoffel symbol and defined as

$$\Gamma^\nu_{\sigma\mu} = \frac{1}{2} g^{\nu\rho} \left(\partial_\sigma g_{\rho\mu} + \partial_\mu g_{\rho\sigma} - \partial_\rho g_{\sigma\mu}\right). \tag{15}$$

4 Energy spectrum of a particle near the proximity Reissner-Nordström black hole

In this section, the Dirac equation, as stated earlier, is applied in order to define the energy spectrum of the fermionic particles around the RN black hole. According to the metric of the RN space-time Eq. (1) the tetrad parameters are defined as follows,

$$e^a_\mu = diag(f^{\frac{1}{2}}, f^{-\frac{1}{2}}, r, r\sin\theta). \tag{16}$$

Furthermore, by considering $A_\mu = \left(\frac{Q}{r}, 0, 0, 0\right)$ (Q is the charge of the black hole) also computing the covariant derivative D_μ as,

$$\begin{aligned} D_0 &= \partial_0 + \frac{iqQ}{r} - \frac{1}{2}f'(r)\gamma_0\gamma_1, \\ D_1 &= \partial_1, \\ D_2 &= \partial_2 - if^{\frac{1}{2}}(r)\gamma_5\gamma_0\gamma_3, \\ D_3 &= \partial_3 + i\sin\theta f^{\frac{1}{2}}(r)\gamma_5\gamma_0\gamma_2 - i\cos\theta\gamma_5\gamma_0\gamma_1, \end{aligned} \tag{17}$$

in to the Dirac equation (Eq. (6)) to determine

$$\begin{aligned} [i\gamma^0 f^{-\frac{1}{2}}(r)\partial_0 + if^{\frac{1}{2}}(r)\gamma^1\partial_1 + \frac{i\gamma^2}{r}\partial_2 + \frac{i\gamma^3}{r\sin\theta}\partial_3 + \frac{i\gamma^1 f^{-\frac{1}{2}}(r)}{2r}(2f(r) + \frac{r}{2}f'(r)) \\ + \frac{i\gamma^2\cot\theta}{2r} - \frac{\gamma^0 f^{-\frac{1}{2}}(r)qQ}{r} - m]\Psi = 0. \end{aligned} \tag{18}$$

Consider the wave function $\Psi = \frac{e^{-iEt}}{rf^{\frac{1}{4}}(r)\sin^{\frac{1}{2}}\theta}\Phi$ [26], in Eq. (18) to define

$$\left[\gamma^0 f^{-\frac{1}{2}}(r)E + i\gamma^1 f^{\frac{1}{2}}(r)\partial_1 + \frac{i\gamma^2}{r}\partial_2 + \frac{i\gamma^3}{r\sin\theta}\partial_3 - \gamma^0 f^{-\frac{1}{2}}(r)\frac{qQ}{r} - m\right]\Phi = 0. \tag{19}$$

The angular parts are contemplated as an operator K [27],

$$K = \gamma^0\gamma^1\left(\gamma^2\partial_2 + \frac{\gamma^3\partial_3}{\sin\theta}\right), \qquad K\Phi = k\Phi, \tag{20}$$

where $k = 0, \pm1, \pm2, ...$ represents the eigenvalues of operator K. Moreover, since the radial part is our concern, thus, the angular part $Z(\theta,\phi)$ and Φ can be expressed in

$$\Phi = Z(\theta,\varphi)\begin{pmatrix} g(r)I_2 \\ ih(r)I_2 \end{pmatrix}, \tag{21}$$

here $I_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then Eq. (19) can be written in two separate, radial forms

$$f^{\frac{1}{2}}\partial_1 h(r) - \frac{k}{r}h(r) + g(r)\left[f^{-\frac{1}{2}}(E - \frac{qQ}{r}) - m\right] = 0, \tag{22}$$

$$f^{\frac{1}{2}}\partial_1 g(r) + \frac{k}{r}g(r) - h(r)\left[f^{-\frac{1}{2}}(E - \frac{qQ}{r}) + m\right] = 0. \tag{23}$$

In order to evaluate the wave function near the outer event horizon ($r \longrightarrow r_+$), let us determine parameters p and y as $p = \left(\frac{r_-}{r_+} - 1\right)^{-\frac{1}{2}}$, $y = \left(1 - \frac{r}{r_+}\right)^{\frac{1}{2}}$, therefore, the Eqs. (22) and (23) can be written as

$$y\frac{\partial h}{\partial y} + 2kyph(r) - 2pr_+\left[p(E - \frac{qQ}{r_+}) - my\right]g(r) = 0, \tag{24}$$

$$y\frac{\partial g}{\partial y} - 2kypg(r) + 2pr_+\left[p(E - \frac{qQ}{r_+}) + my\right]h(r) = 0. \tag{25}$$

For the massless fermionic particles $m = 0$, the answers to the above equations will be defined as following [28],

$$g(y) = e^{-2kpy}y^{i\alpha}L_n^{\alpha}(4ikpy), \tag{26}$$

and

$$h(y) = e^{-2kpy}y^{i\gamma}L_s^{\gamma}(4kpy), \tag{27}$$

where parameters α, n, γ and s are defined as follows

$$\alpha = \frac{2}{\omega}\left(E - \frac{qQ}{r_+}\right) \quad , \quad n = \frac{1}{\omega}\left(E - \frac{qQ}{r_+}\right) - \frac{1}{2} + i, \tag{28}$$

and

$$\gamma = \frac{2i}{\omega}\left(E - \frac{qQ}{r_+}\right) \quad , \quad s = \frac{i}{\omega}\left(E - \frac{qQ}{r_+}\right), \tag{29}$$

which $\omega = \frac{r_+ - r_-}{2r_+^2}$, represents frequency. Hence we choose the wave function that decreases at a constant damping rate, so the energy spectrums Eqs. (28) and (29) are considered as particle energy near the Reissner-Nordström black hole. Then use energy to obtain the entropy and thermal properties of the black hole.

5 Thermal properties of Reissner-Nordström black hole with energy spectrum

To investigate the thermal properties such as energy, heat capacity, and entropy of RN black hole, initially, the partition function is utilized [29]. The partition function, which represents an exponential function, is the amount of all possible energies of that system and expressed by

$$Z = \frac{\zeta^N}{N!}, \tag{30}$$

we can interpret that ζ^N over-counts the energy levels in which all N particles are in different states by exactly $N!$. Now let us consider that, N indicates the number of particles near the Reissner-Nordström black hole, therefore we are able to compute the partition function for N indefinite particles in the canonical set by applying the energy spectrum (28) to obtain:

$$\zeta = \sum_{n=0}^{\infty} e^{-\beta E_n} = e^{\frac{-\beta qQ}{r_+}} e^{i\beta\omega} \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})\omega}, \tag{31}$$

where β mentions the inverse of temperature ($\beta = \frac{1}{T}$). By considering the real term of $e^{i\beta\omega} = \cos(\beta\omega) + i\sin(\beta\omega)$, then Eq. (31) can be written as

$$\zeta = \frac{e^{-\frac{\beta\omega}{2}}}{-e^{-\beta\omega}+1} e^{\frac{-\beta qQ}{r_+}} \cos(\beta\omega). \tag{32}$$

In order to define the average energy, the equation which represents the relation between energy and partition function is utilized by

$$< E >= -\frac{\partial \ln Z}{\partial \beta}. \tag{33}$$

Substituting Eq. (32) in Eq. (30) then plugging in Eq. (33) to approach the following average value of energy,

$$< E >= \frac{N\omega}{2} + \frac{N\omega e^{-\beta\omega}}{1-e^{-\beta\omega}} + \frac{NqQ}{r_+} + N\omega\tan(\beta\omega). \tag{34}$$

The behavior of the energy in Eq. (32) with respect to inverse temperature or β is illustrated in Fig. (1), as it is shown, by increasing the number of particles N energy also rise, moreover, the action is same for various values of $\omega = 0.2$ (left) and $\omega = 0.04$ (right). Furthermore,

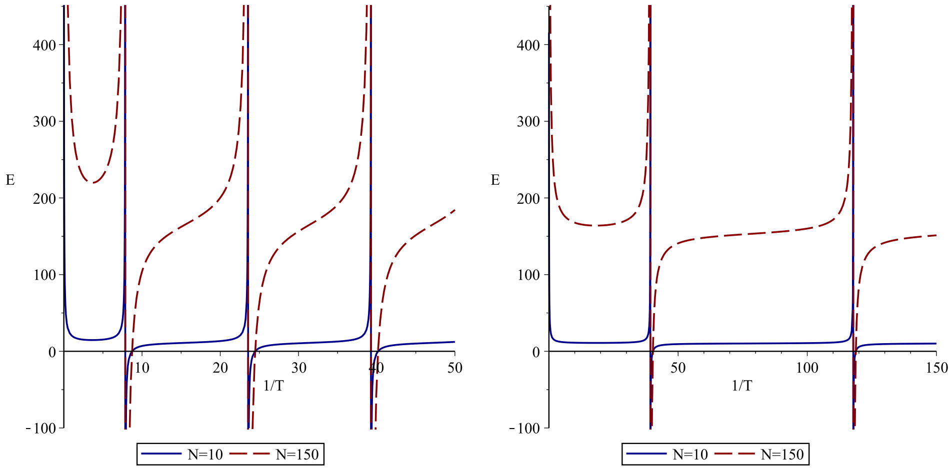

Figure 1: Graph of energy E Vs $\beta = \frac{1}{T}$ for Dirac particles in various particle number N, $\omega = 0.2$ (left) and $\omega = 0.04$ (right). The physical parameters are selected as $M = q = Q = 1$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article [https://jhap.du.ac.ir/].)

we can also obtain the heat capacity of the RN black hole by using the average energy by the following formula,

$$C_V = \frac{\partial < E >}{\partial T}. \tag{35}$$

Plugging Eq. (34) in Eq. (35) to reach

$$C_V = N\omega^2 \frac{1}{T^2} \left[\frac{e^{-\frac{1}{T}\omega}}{(1-e^{-\frac{1}{T}\omega})^2} - \frac{1}{\cos^2(\frac{1}{T}\omega)} \right]. \tag{36}$$

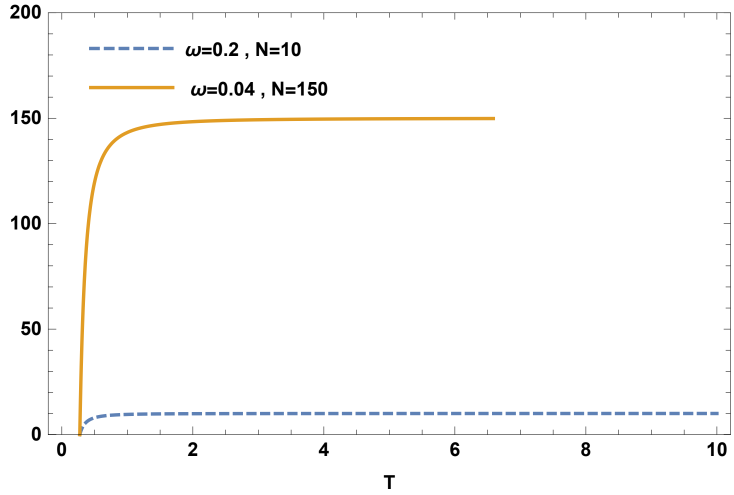

Figure 2: The heat capacity in terms of temperature. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article [https://jhap.du.ac.ir/].)

As shown in Fig. (2), at high temperatures, the heat capacity is equal to the constant number of particles.
We obtain the entropy by using the average energy and the partition function as [29]:

$$S = \ln Z + \beta < E > . \tag{37}$$

Now, we can define the entropy in the RN background, with considering Stirling's approximation as ($\ln N! = N \ln N - N$) and utilizing the above expressions for Z and E as

$$S = N\beta\omega \frac{e^{-\beta\omega}}{1 - e^{-\beta\omega}} - N \ln(1 - e^{-\beta\omega}) + N \ln(\cos\beta\omega) + N\beta\omega \tan(\beta\omega) - N \ln N + N. \tag{38}$$

Fig. 3 shows the behavior of entropy S of Dirac particles in RN background under varying parameters N and ω.

6 Holography and information theory

Since the boundary of the curve in the RN black hole plays an important role in holography and AdS/CFT, thus, one can think about the boundary of the RN black hole as a storage device for information. Assuming that the holographic principle holds, the maximal storage space, or a total number of bits, is proportional to the area A. Let us denote the number of used bits by N_0. Therefore, it is possible to assume that this number will be proportional to the area as well [30]. We know from the holographic principle that all information about an object is stored on its surface. It also states that a bit of information is stored in an area of the Planck length. For the black hole, its information is stored on its boundary, and we use the event horizon area to describe its feature.
In this section, the aim is to figure out the way of information storing by particles in the RN space-time on the surface of the event horizon. First, to determine the maximum entropy, the derivative of Eq. (38) with respect to N is defined, $\frac{\partial S}{\partial N} = 0$ as follows,

$$S_{Max} = N. \tag{39}$$

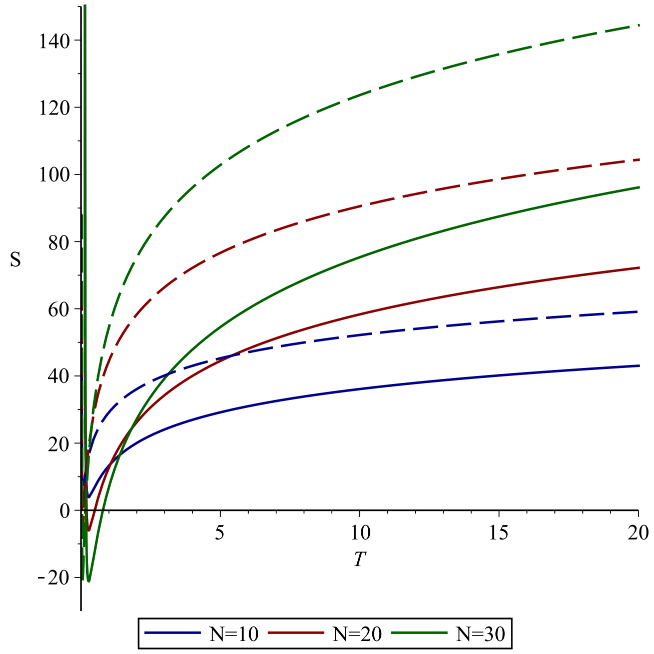

Figure 3: Graph of entropy S versus temperature T for various values of particle number N and frequency ω. The solid lines mention $\omega = 0.2$ and dashed line indicate $\omega = 0.04$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article [https://jhap.du.ac.ir/].)

Given the particles near the event horizon, we can consider the number of particles proportional to the surface of the black hole

$$A = \tilde{q}N, \tag{40}$$

where $\tilde{q}$ is a constant value. Also, we can set the maximum entropy equal to the Bekenstein entropy $S_{Max} = S_{BH}$ and $S_{BH} = \frac{A}{4}$ [14, 31]. In this case, according to the Eqs. (39) and (40), we can define $\tilde{q} = 4$. Based on Verlinde's theory [30], each surface can be considered as information bits, so we consider the surface of the black hole as a set of information bits and write its relation as follows:

$$A = N_0. \tag{41}$$

Given the Eqs. (40) and (41), the relation between the particle number and the information bits will be obtained as

$$N_0 = 4N. \tag{42}$$

The Eq. (42) reveals that each particle in a near the RN black hole has 4 bit of information.

7 Discussion and result

In this article, we examined the behavior of the Dirac particle the same as our previous investigation [22] for Klein-Gordon particles. It is worth mentioning which for Klein Gordon particles, the energy obtained was real and in the form of a harmonic oscillator, but for

Dirac particles, the energy derived as both real and imaginary, and the real part was in the form of a harmonic oscillator plus an additional term. We used holography and found that each Klein Gordon particle contained eight bits of information on the surface of the RN space-time, while the Dirac particle contained four bits of information. One of the factors that make the Dirac particle less information than the Klein-Gordon particle can be the principle of Paula's exclusion. In this study, we have derived the thermodynamical properties of Dirac particles with the regarded figures from giving a wider perspective on RN background.

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