Journal of Holography Applications in Physics
Volume 2, Issue 2, Spring 2022, 57-64
© Available online at http://jhap.du.ac.ir
DOI: 10.22128/jhap.2022.521.1021

Journal of Holography Applications in Physics

Regular article

Holographic description of shear viscosity to entropy ratio

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Received: March 09, 2022; Revised: April 15, 2022; Accepted: April 29, 2022

Abstract. Strongly coupled field theory can be described by the calculation of $\frac{\eta}{s}$ from fluid-gravity duality since this value is proportional to the inverse square coupling of field theory. This ratio should be larger than $\frac{\eta}{s} \geq \frac{1}{4\pi}$, and it is known as a conjecture in the literature. We calculate the ratio of shear viscosity per entropy density for a dilaton black brane in AdS space-time. Our result shows that this bound is saturated in this black brane and the field theory dual of our model is the same as the field theory dual Einstein gravity.

Keywords: Shear Viscosity; Entropy Density; Fluid/Gravity Duality.

1 Introduction

AdS/CFT duality introduced by Maldacena [1] relates two kinds of theories: gravity in (n+1)-dimension and field theory in n-dimension. The most familiar example, the AdS/CFT duality asserts that SYM $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory is dual to Type IIB string theory on $AdS_5 \times S^5$. There is no way to solve the strongly coupled field theories either analytically or perturbatively. AdS/CFT duality is a technique to overcome this problem. By using this duality, we can translate the strongly coupled field theory into a weakly gravitational theory and vice versa. The map between these two different theories is known as a holographic dictionary. In the long-wavelength limit, this duality leads to fluid/gravity duality. Any fluid is characterized by some transport coefficients. These coefficients identify the underlying microscopic properties of fluids, which in turn are rooted in the field theory interactions at strong coupling. So, the gauge/gravity duality would be a proper tool to calculate these coefficients. In this work, our interest is the shear viscosity, one of the transport coefficients. The conservation of energy and momentum in relativistic Hydrodynamics is as follows,

$$\nabla_\mu T^{\mu\nu} = 0 \qquad (1)$$
$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu} \qquad (2)$$

Note that the term “relativistic fluid” doesn't mean the fluid necessarily moves near the speed of light. However, the Lorenz symmetry preserves the relativistic fluid.

We introduce a parameter expansion $\varepsilon = \frac{\ell_{mfp}}{L}$, where ℓ_{mfp} and L are the mean free path and the characterized length of the system or the scale for the field fluctuations, respectively. The scale of field variations has to be large compared to the mean free path, $\ell_{mfp} \ll L$ for the validity of hydrodynamics regime on the boundary. We know that the regime where the fluid is valid corresponds to a theory with large AdS black holes. We can expand the energy-momentum tensor in terms of ε when it is $\varepsilon \ll 1$ [1-3]

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} + pg^{\mu\nu} - \sigma^{\mu\nu} \tag{3}$$

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}[\eta\left(\partial_{\alpha}u_{\beta} + \partial_{\beta}u_{\alpha} - \frac{2}{3}g_{\alpha\beta}\partial_{l}u^{l}\right) + \xi g_{\alpha\beta}\partial_{l}u^{l} \tag{4}$$

where η and ξ are shear and bulk viscosities, respectively.
In this article, we calculate shear viscosity by Green-Kubo formula,

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}ImG^{R}_{ij,ij}(\omega, 0), \tag{5}$$

where $G^{R}_{ij,ij}(\omega, 0)$is as follows,

$$G^{R}_{ij,ij}(\omega, 0) = \int dtd\boldsymbol{x}e^{i\omega t}\theta(t)\langle[T_{ij}(t, \boldsymbol{x}), T_{ij}(0, \boldsymbol{O})]\rangle \tag{6}$$

The ratio of shear viscosity per entropy density is proportional to the inverse square coupling of quantum thermal field theory, $\frac{\boldsymbol{\eta}}{\boldsymbol{s}} \sim \frac{\boldsymbol{1}}{\boldsymbol{\lambda^2}}$, where $\boldsymbol{\lambda}$ is the coupling constant of field theory. In particular, the stronger the coupling, the weaker the shear viscosity per entropy density. In theories with gravity duals, even in the limit of infinite coupling the ratio $\frac{\boldsymbol{\eta}}{\boldsymbol{s}}$ cannot be made smaller than $\frac{1}{4\pi}$.

In the following section, we review the dilaton black brane in AdS space-time. Then calculate the shear viscosity to the entropy density ratio and find out that it satisfies the conjectured bound $\frac{1}{4\pi}$.

2 Dilaton black brane solution

We consider the 5-dimensional theory in which gravity is coupled to dilaton and Maxwell field with an action [4],

$$S = \int d^5x\sqrt{-g}(R - 2\Lambda - \frac{4}{3}\partial_{\mu}\phi\partial^{\mu}\phi - V(\phi) - e^{-\frac{4\alpha\phi}{3}}F^2), \tag{7}$$

where R is the Ricci scalar, $F^2 = F_{\mu\nu} F^{\mu\nu}$ is the usual Maxwell contribution, and $V(\phi)$ is a potential of dilaton ϕ, which is with respect to the cosmological constant. α is an arbitrary constant governing the strength of the coupling between the dilaton and the Maxwell field,

$$V(\phi) = \frac{\Lambda}{3(2+\alpha^2)^2}[-12\alpha^2(1-\alpha^2)e^{-\frac{8(\phi-\phi_0)}{3\alpha}} + 12(4-\alpha^2)e^{-\frac{4\alpha(\phi-\phi_0)}{3}} + 72\alpha^2 e^{-\frac{2(\phi-\phi_0)(2-\alpha^2)}{3\alpha}}] \quad (8)$$

Varying the action with respect to the metric, Maxwell, and dilaton fields, respectively yield,

$$R_{\mu\nu} = \frac{4}{3}\left(\partial_\mu\phi\partial_\nu\phi + \frac{1}{4}g_{\mu\nu}\right) + 2e^{\frac{-4\alpha\phi}{3}}\left(F_{\mu\alpha}F_\nu^\alpha - \frac{1}{6}g_{\mu\nu}F^2\right) \quad (9)$$

$$\partial_\mu\left(\sqrt{-g}\, e^{\frac{-4\alpha\phi}{3}} F^{\mu\nu}\right) = 0 \quad (10)$$

$$\partial_\mu\partial^\mu\phi = \frac{3}{8}\frac{\partial V}{\partial\phi} - \frac{\alpha}{2}e^{\frac{-4\alpha\phi}{3}}F^2 \quad (11)$$

The metric for the well-known 5-dimensional dilaton black hole with the cosmological constant is given by:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)\left[1-\left(\frac{r_-}{r}\right)^2\right]^{\frac{\alpha^2}{\alpha^2+2}}}dr^2 + r^2\left[1-\left(\frac{r_-}{r}\right)^2\right]^{\frac{\alpha^2}{\alpha^2+2}}d\Omega_3^2, \quad (12)$$

where

$$f(r) = \left[1-\left(\frac{r_+}{r}\right)^2\right]\left[1-\left(\frac{r_-}{r}\right)^2\right]^{\frac{2-\alpha^2}{2+\alpha^2}} - \frac{1}{3}\Lambda r^2\left[1-\left(\frac{r_-}{r}\right)^2\right]^{\frac{\alpha^2}{\alpha^2+2}}, \quad (13)$$

and

$$d\Omega_3^2 = d\theta^2 + \sin^2(\theta)\, d\phi^2 + \sin^2(\theta)\sin^2(\phi)\, d\phi^2, \quad (14)$$

If the solid angle is small, we have black brane,

$$d\Omega_3^2 = \frac{1}{l^2}\left(dx_1^2 + dx_2^2 + dx_3^2\right) = \frac{1}{l^2}d\vec{x}^2 \quad (15)$$

Notice r is the radial coordinate that puts us from bulk to boundary. In the following, we apply dimensionless variable u instead of r, that is $u = (\frac{b}{r})^2$, then

$$ds^2 = \frac{b^2}{u}\left(1-\frac{a^2}{b^2}u\right)^{\frac{\alpha^2}{\alpha^2+2}}[-f(u)dt^2 + d\vec{x}^2] + \frac{du^2}{4u^2 f(u)\left(1-\frac{a^2}{b^2}u\right)^{\frac{\alpha^2}{\alpha^2+2}}},$$

$$ds^2 = g_{uu}dt^2 + g_{\mu\nu}dx^\mu dx^\nu = g_{MN}dx^M dx^N, \qquad (16)$$

where

$$f(u) = -\left(\frac{u}{b^2}(1-u)\left(1-\frac{a^2}{b^2}u\right)^{\frac{2-2\alpha^2}{2+2\alpha^2}} - \frac{2}{l^2}\right), \qquad (17)$$

where $\mu,\nu = 0,..,3$, $M,N = 0,..,4$, $M,N = 0,..,4$. $r_+ = b$ and $r_- = a$ are the event horizons. l is the radius space-time.

3 $\frac{\eta}{s}$ for dilaton black brane solution

For the calculation of shear viscosity, we perturbed the background metric as $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$ [5-8]. Considering the abbreviation $h_{\mu\nu} \equiv \phi$, the mode equation is found to be,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi(t,u,\vec{x})\right) = 0 \qquad (18)$$

By applying Fourier transformation to $(t,\vec{x})$ coordinates in Eq. (15) and setting the momentum to zero in Green-Kubo formula, then introducing $\phi(t,u,\vec{x}) = G(u)\phi_0(t,\vec{x})$ where content $\phi_0(t,\vec{x})$ is the source for both graviton in the bulk and the stress tensor on the boundary, we will get,

$$\frac{d^2G(u)}{du^2} + \frac{1}{2}\left(\frac{H'(u)}{H(u)} + \frac{F'(u)}{F(u)} - \frac{2}{u} + \frac{3B'(u)}{B(u)}\right)\frac{dG(u)}{du} + \frac{\ell^2\omega^2 B(u) - k^2 H(u)}{4\,u\,r_0^2\,F(u)H(u)B(u)}G(u) = 0, \qquad (19)$$

with $F'(u) = \frac{dF(u)}{du}$ and $H'(u) = \frac{dH(u)}{du}$.

The long-wavelength dynamics of the strongly coupled field at the boundary can be described in terms of the near horizon data of the black brane solution in the bulk space-time. Therefore, we solve the mode equation close to the horizon,

$$H(u) \approx -(1-u)H'(1) \qquad (20)$$

$$F(u) \approx -(1-u)F'(1) \qquad (21)$$

$$F(u)H(u) \approx (1-u)^2F'(1)H'(1) = (1-u)^2\left(\frac{2\pi\ell^2 T}{r_0}\right)^2 \qquad (22)$$

Substituting Eq. (20) - Eq. (22) into the mode equation Eq. (19) gives us,

$$\frac{d^2\,G(u)}{du^2} - \frac{1}{1-u}\frac{dG(u)}{du} + \frac{\omega^2}{16\,\pi^2 T^2\,(1-u)^2}G(u) = 0 \tag{23}$$

The above equation has a solution in the form of $G(u) = (1-u)^{\beta}$. By putting this ansatz into the Eq. (19), we can obtain β,

$$\beta = \pm\frac{I\varpi}{2}, \qquad \varpi = \frac{\omega}{2\pi T} \tag{24}$$

Retarded Green's function on the boundary corresponds to the ingoing mode of the near horizon. Due to event horizon properties, the outgoing mode doesn't exist. By putting the outgoing solution aside, we will have,

$$G(u) = (1-u)^{-\frac{I\varpi}{2}} \tag{25}$$

Here we consider the following ansatz for the mode equation Eq. (21),

$$G(u) = \tilde{F}(u)^{-\frac{I\varpi}{2}}\left(\tilde{h}_0(u) + \frac{I\varpi}{2}\tilde{h}_1(u) + O(\varpi^2)\right), \tag{26}$$

where $\tilde{F}(u) = \sqrt{F(u)H(u)}$ Since we want to normalize $G(u)$ on the boundary, we choose $\tilde{h}_0(u) = 1$.

For determining $\tilde{h}_1(u)$ we plug (26) in (19) and keep to first order of ϖ,

$$\tilde{h}_1{}'' + \left(\frac{\tilde{F}'(u)}{\tilde{F}(u)} - \frac{1}{u} + \frac{3B'(u}{B(u)}\right)\tilde{h}_1{}' - \frac{\tilde{F}''}{\tilde{F}} + \frac{\tilde{F}''}{\tilde{F}}\left(\frac{1}{u} - \frac{3B'(u)}{B(u)}\right) = 0 \tag{27}$$

It can be easily solved to find,

$$\frac{\tilde{F}\tilde{h}_1{}' - \tilde{F}'}{u\,B(u)^{\frac{-3}{2}}} = C_1 \tag{28}$$

$$\tilde{h}_1 = log\frac{\tilde{F}}{C_2} + C_1\int_b^u \frac{n\,B(n)^{\frac{-3}{2}}}{\tilde{F}(n)}dn, \tag{29}$$

where C_1 and C_2 are integration constants. For our purposes, the explicit form of $\tilde{h}_1$ is not important. It would be enough to find C_1 by demanding $\tilde{h}_1$ to be nonsingular at the horizon. So we may investigate the near horizon behavior of the integral in (29) as follows,

$$\tilde{F} \approx -(1-u)\tilde{F}'(1) = -(1-u)\frac{2\pi l^2 T}{b} \quad (30)$$

$$\tilde{h}_1 \approx log\frac{1-u}{C_2} - \frac{C_1 B(u=1)^{\frac{-3}{2}} b}{2\pi l^2 T} log\,(1-u) \quad (31)$$

To have non-singular $\tilde{h}_1$ at the horizon, C_1 is chosen to be,

$$C_1 = \frac{2\pi l^2 T}{b} B(u=1)^{\frac{3}{2}} \quad (32)$$

The prescription for calculation of retarded Green's function is presented by Son [5-7]. We calculate retarded Green's function by this prescription as follows:

$$G^R(x-y) = -\sqrt{-g} g^{uu} G^*(u)\partial_u G(u)|_{u\to 0} = \frac{I\ \omega\ b^4}{\pi l^5 T}\left(\frac{\tilde{F}' - \tilde{F}\tilde{h}_1{}'}{u\ B(u)^{\frac{-3}{2}}}\right)|_{u\to 0}$$
$$= -\frac{I\ b^4 \omega}{\pi l^5 T} C_1 = -\frac{I\ b^3 \omega}{l^3} g(u=1)^{\frac{3}{2}} \quad (33)$$

Now we can calculate shear viscosity by using the Green-Kubo formula,

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega} Im G_{yy}^{xx}(\omega,\vec{0}) = \frac{b^3}{l^3} g(u=1)^{\frac{3}{2}} \quad (34)$$

The entropy can be found by using the following Hawking-Bekenstein formula,

$$S = \frac{A}{4G} = \frac{b^3\ V_3}{4\ G\ l^3} g(u=1)^{\frac{3}{2}} \quad (35)$$

The entropy density,

$$s = \frac{S}{V_3} = \frac{A}{4G} = \frac{b^3}{4\ G\ l^3} g(u=1)^{\frac{3}{2}} \quad (36)$$

where V_3 is the volume of the constant t and r hyper-surface with radius r_0 and in the last line, we used $\frac{1}{16\pi G} = 1$, so $\frac{1}{4\pi} = 4G$.

Then the ratio of shear viscosity to entropy density is,

$$\frac{\eta}{s} = \frac{1}{4\pi} \quad (37)$$

4 Results and Discussion

We showed that the lower bound of the $\frac{\eta}{s}$ preserves for Dilaton black brane. This bound is known as the KSS conjecture [6] and is considered for strongly interacting systems where a reliable theoretical estimate of the viscosity is not available. It tells us that the $\frac{\eta}{s}$ has a lower bound, $\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}$, for all relativistic quantum field theories at finite temperature without chemical potential and can be interpreted as the Heisenberg uncertainty principle [5]. This conjecture violates for higher derivative gravities like the Gauss-Bonnet gravity [8]. The ratio of shear viscosity per entropy density is proportional to the inverse square coupling of quantum thermal field theory, $\frac{\eta}{s} \sim \frac{1}{\lambda^2}$, where λ is the coupling constant of field theory. In particular, the stronger the coupling, the weaker the shear viscosity per entropy density. In theories with gravity duals, even in the limit of infinite coupling the ratio $\frac{\eta}{s}$ cannot be made smaller than $\frac{1}{4\pi}$. Therefore, the dual of Dilaton black brane is the same as Schwarzschild black brane.

5 Conclusion

We showed that KSS bound is saturated for Dilation black brane, and the coupling of field theory dual of our model and Schwarzschild black brane is the same.

Acknowledgment

The author would like to thank Shahrokh Parvizi and the referee of JHAP for useful comments and suggestions.

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