Journal of Holography Applications in Physics
Volume 2, Issue 2, Spring 2022, 1-14
©Available online at http://jhap.du.ac.ir
DOI:10.22128/jhap.2022.488.1015

**Journal of Holography Applications in Physics**

**Regular article**

# Entanglement entropy in Horndeski gravity

**Fabiano. F. Santos**
Instituto de Física, Universidade Federal do Rio de Janeiro, 21941-972 Rio de Janeiro, Brazil.
email: fabiano.ffs23@gmail.com

Received: January 11, 2022; Revised: February 09, 2022; Accepted: February 23, 2022.

**Abstract.** In this work, we explore the holographic entanglement entropy with an infinite strip region of the boundary in Horndeski gravity. In our prescription, we consider the spherical and planar topologies black holes in the $AdS_4/CFT_3$ scenario. In such a framework, we show the behavior of the entanglement entropy in the function of the Horndeski parameters. Such parameters modify the information store of subsystem A, especially when the parameter $\gamma$ increases the information about the subsystem will also increase or decrease when it decreases. Thus, with this scheme we compute the first law of entanglement thermodynamics in Horndeski gravity and we show that a very small subsystem obeys the analogous property of the first law of thermodynamics if we excite the system.

*Keywords*: Holographic Entanglement; Horndeski Gravity; Field Theory.

## 1 Introduction

In recent years, the scenario of entanglement entropy have call attention in special the works of [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] where the $S_A$ as the entropy for an observer accessible only to the $A$ subsystem and it cannot receive any $B$ signal. Such subsystem $B$ is analogous to the interior of the black hole horizon. For an observer who is sitting on $A$, that is, outside the horizon. On the other hand, because this analogy is not correct, the quantum correction involving a loop for the entropy BH in the presence of fields of the matter is known to be equal to the entanglement entropy [9]. Thus, we have that this interesting relation provides us with an important tip to find the holographic dual of the entanglement entropy. In the context of the $AdS_3/CFT_2$ correspondence presented by [1, 5, 7] is possible to calculate the entropy $S_A$ in a $CFT_2$, holographically — See Fig. 1. This entropy is calculated as follows:

$$S_A = Min_{\Sigma_A}\left[\frac{Area(\Sigma_A)}{4G_N}\right] \tag{1}$$

Note that $\Sigma_A$ represents a two-dimensional surface, that is, four-dimensional in $AdS_3$ satisfying $\partial\Sigma_A = \partial A$ since $\Sigma_A$ is a counterpart to $A$. In addition to these considerations, we have that (1) is obtained for all surfaces, $\Sigma_A$, which is a minimal surface. The equation (1) is applied to any static configuration. Besides, as we know, the minimum surface surrounding the area is well defined in the static case, thus making it possible to consider Euclidean AdS space equivalently. It has an interesting feature that is a striking similarity to the Bekenstein-Hawking (BH) entropy of black holes [1, 5, 7, 8, 9]. Besides, as we know, the minimum surface surrounding the area is well defined in the static case, thus making

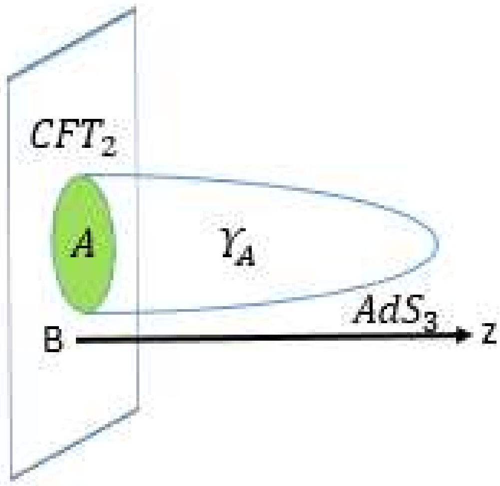


Figure 1: The calculation of holographic entanglement entropy.

it possible to consider Euclidean AdS space equivalently. Motivated by the applications of [10], we propose a scenario of entanglement entropy in Horndeski gravity where is considered an analytically the entanglement entropy of the subsystem $A$ in the $(2+1)$-dimensional boundary field theory. In our case we consider the duality $\mathrm{AdS}_4/\mathrm{CFT}_3$, where on the gravity side, we have a planar black hole solution in Horndeski gravity, and with this solution, we extract the length and the area integral to the subsystem $A$.

The main motivation to investigate the play role of Horndeski gravity in the entanglement entropy scenario is due the recent investigations of the AdS/CFT correspondence in this gravity [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Besides, beyond to the classes of boundary field theories that were studied by Ryu and Takayanagi [7, 8] other classes were proposed using this conjecture, such as AdS black holes with dual charge in the bulk [11, 12, 13]. But in our case, we compute the entanglement entropy in $\mathrm{AdS}_4/\mathrm{CFT}_3$ within Horndeski gravity for spherically and planar black holes. For this boundary field theory at finite temperature, we address the issues related to entanglement thermodynamics within Horndeski gravity in our setup, for more discussion about this, see [14, 17, 18, 10]. For the available excited states, we compute the stress-energy tensor of boundary field theory in Horndeski gravity following the prescription of [21, 31].

This work is summarized as follows. In Sec. 2 we address the issue of finding black hole solutions in Horndeski gravity. In Sec. 3, we present the entanglement entropy. In Sec. 4 we compute the entanglement thermodynamics in Horndeski gravity. In section Sec. 4, we present our conclusions.

## 2 Black hole solutions in Horndeski gravity

In this section we address the issue of finding black hole solutions in Horndeski gravity [19, 20, 21, 32, 33, 34, 35, 36]. Black holes in Horndeski's theory have been previously studied in [19, 20, 21, 33, 34, 36]. The Horndeski Lagrangian is given by

$$\mathcal{L}_H = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5, \tag{2}$$

$$\mathcal{L}_2 = G_2(X,\phi), \tag{3}$$
$$\mathcal{L}_3 = -G_3(X,\phi)\Box\phi, \tag{4}$$
$$\mathcal{L}_4 = G_4(X,\phi)R + \partial_X G_4(X,\phi)\delta^{\mu\nu}_{\alpha\beta}\nabla^{\alpha}_{\mu}\phi\nabla^{\beta}_{\nu}\phi, \tag{5}$$
$$\mathcal{L}_5 = G_5(X,\phi)G_{\mu\nu}\nabla^{\mu}\nabla^{\nu}\phi$$
$$-\frac{1}{6}\partial_X G_5(X,\phi)\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\nabla^{\alpha}_{\mu}\phi\nabla^{\beta}_{\nu}\phi\nabla^{\gamma}_{\rho}\phi, \tag{6}$$

where $X \equiv -\frac{1}{2}\nabla_{\mu}\phi\nabla^{\nu}\phi$. Furthermore, an interesting special truncation of this theory was presented by [37, 38, 39, 40],where the idea is realize a constrain the coefficients $G_k(X,\phi)$. Through this truncation and considering the non-minimal kinetic coupling, we have

$$I[g_{\mu\nu},\phi] = \int\sqrt{-g}d^4x\mathcal{L}, \tag{7}$$
$$\mathcal{L} = \kappa(R-2\Lambda) - \frac{1}{2}(\alpha g_{\mu\nu} - \gamma G_{\mu\nu})\nabla^{\mu}\phi\nabla^{\nu}\phi.$$

Here we used $\kappa = (16\pi G)^{-1}$ in the action (7). Such action have a non-minimal scalar-tensor coupling and we can define a new field $\phi^{'} \equiv \psi$. The field has a dimension of $(mass)^2$ controlled by the parameters $\alpha$ and $\gamma$ where $\alpha$ is dimensionless and $\gamma$ has a dimension of $(mass)^{-2}$. The equations of motion are:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\kappa}T_{\mu\nu}, \tag{8}$$

where $T_{\mu\nu} = \alpha T^{(1)}_{\mu\nu} + \gamma T^{(2)}_{\mu\nu}$. The energy-momentum tensors $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$ take the following form

$$T^{(1)}_{\mu\nu} = \nabla_{\mu}\phi\nabla_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\nabla_{\lambda}\phi\nabla^{\lambda}\phi$$
$$T^{(2)}_{\mu\nu} = \frac{1}{2}\nabla_{\mu}\phi\nabla_{\nu}\phi R - 2\nabla_{\lambda}\phi\nabla_{(\mu}\phi R^{\lambda}_{\nu)} - \nabla^{\lambda}\phi\nabla^{\rho}\phi R_{\mu\lambda\nu\rho}$$
$$-g_{\mu\nu}\left[-\frac{1}{2}(\nabla^{\lambda}\nabla^{\rho}\phi)(\nabla_{\lambda}\nabla_{\rho}\phi) + \frac{1}{2}(\Box\phi)^2 - (\nabla_{\lambda}\phi\nabla_{\rho}\phi)R^{\lambda\rho}\right]$$
$$-(\nabla_{\mu}\nabla^{\lambda}\phi)(\nabla_{\nu}\nabla_{\lambda}\phi) + (\nabla_{\mu}\nabla_{\nu}\phi)\Box\phi + \frac{1}{2}G_{\mu\nu}(\nabla\phi)^2. \tag{9}$$

And the scalar field equation is given by

$$\nabla_{\mu}[(\alpha g^{\mu\nu} - \gamma G^{\mu\nu})\nabla_{\nu}\phi] = 0. \tag{10}$$

In our case for Einstein-Horndeski gravity, we consider the following *Ansatz* for a four-dimensional black hole of the form

$$ds^2 = \mathcal{R}^2\left(-r^2f(r)dt^2 + r^2(dx^2+dy^2) + \frac{dr^2}{r^2f(r)}\right). \tag{11}$$

Now, following the results of [19, 20, 21, 33, 34, 36], we can find the black hole solution through the imposing of the radial component with the conserved current that vanishes identically without restricting the radial dependence of the scalar field [41, 42, 43]:

$$\alpha g_{rr} - \gamma G_{rr} = 0. \tag{12}$$

Taking $\phi'(r) \equiv \psi(r)$ and we can easily note that this condition annihilates $\psi^2(r)$ regardless of its behavior at the horizon. Now, using the equation (12) the metric function $f(r)$ can be found as following:

$$f(r) = \frac{\alpha\mathcal{R}^2}{3\gamma} - \left(\frac{r_h}{r}\right)^3, \tag{13}$$

$$\psi^2(r) = -\frac{2\mathcal{R}^2\kappa(\alpha+\gamma\Lambda)}{\alpha\gamma r^2 f(r)}, \tag{14}$$

The equation of motion (8) are satisfied by these equations. The solution (13) corresponds to black hole solution for asymptotically $AdS_4$ spacetime [36]. Beyond, through $\Lambda = -3/\mathcal{R}^2$ and considering the analyses of [19, 20, 21] in our prescription, we can write the black hole solution

$$f(r) = -\frac{\alpha}{\gamma\Lambda} - \left(\frac{r_h}{r}\right)^3, \tag{15}$$

$$\psi^2(r) = \frac{6\kappa(\alpha+\gamma\Lambda)}{\alpha\gamma\Lambda r^2 f(r)}, \tag{16}$$

we can note that the parameters are defined in the range $-\infty < \alpha/\gamma\Lambda \leq -1$, with $\alpha, \gamma < 0$, or $-1 \leq \alpha/\gamma\Lambda < 0$, with $\alpha, \gamma > 0$. Furthermore, we have that the surface located at $r = r_h$ is infinitely shifted to red in relation to an asymptotic observer. On the other hand, we can see by looking at the equation (16), which $(\alpha + \gamma\Lambda) > 0$, indicates ghost freedom. Thus, stability requires that $(\alpha + \gamma\Lambda)$ is not negative, this leads to an interval of the form $-\infty < \gamma \leq \alpha/(-\Lambda)$. Thus, following these transformation in the metric (11) with solution (15), we can write:

$$f(r) \to -\frac{\alpha}{\gamma\Lambda} f(r), \quad r_h^3 \to -\frac{\alpha}{\gamma\Lambda}\, r_h^3,$$
$$\mathcal{R} \to \left(-\frac{\alpha}{\gamma\Lambda}\right)^{1/2} \mathcal{R}, \quad t \to -\frac{\gamma\Lambda}{\alpha} t,$$
$$x \to \left(-\frac{\gamma\Lambda}{\alpha}\right)^{1/2} x, \quad y \to \left(-\frac{\gamma\Lambda}{\alpha}\right)^{1/2} y, \tag{17}$$

in order to put the black hole solution in the standard form

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^3, \tag{18}$$

$$\psi^2(r) = -\frac{6\kappa(\alpha+\gamma\Lambda)}{\alpha^2 r^2 f(r)}. \tag{19}$$

Using the limit $\psi^2(r \to \infty) = 0$ in the action (7), provided that this is a genuine vacuum solution. The equations (18)-(19), provide that the black hole geometry is regular everywhere (except at the central singularity), the scalar field derivative $\psi(r)$ diverges at horizon [36, 27, 43], but the scalar field does not explode at horizon since it approaches a constant near the horizon as:

$$\phi^2(r) \sim \Big((2\Lambda\mathcal{R}^2(\alpha+\gamma\Lambda)/\alpha^2 r_h^2 f'(r_h))(r - r_h)\Big) + const. \tag{20}$$

Thus, we agree with the no-hair theorem, such discussions were presented by [43]. An interesting characterize is that the scalar field equation (19) is a real function outside the horizon for $r > r_h$, namely, $f(r > r_h) > 0$, and the scalar field is real in the interval $-1 < \alpha/\gamma\Lambda < 0$, with $\alpha, \gamma > 0$. Besides of this analyzes, we have that in the infinity, the scalar field itself diverges as $\phi(r) \sim \ln r$, but not its derivatives $\psi$ that are the ones present in action (7), which are finite at asymptotic infinity [43].

In fact, we have that the appearance of a black hole, which has a flat horizon $\Re^2$, leads us to a physics in IR that corresponds to placing the invariant theory of scale at a finite temperature. So, we have that the temperature of the black hole is given by

$$T(r_h) = \frac{f^{'}(r = r_h)}{4\pi} = \frac{3}{4\pi r_h} \tag{21}$$

## 3 Holographic entanglement entropy in Horndeski gravity

In this section, we present the computations of the entanglement entropy to a subsystem $A$ in Horndeski gravity following the procedures of [1, 5, 7, 8, 9]. For this, considering the metric (11) the three-dimensional CFT lives in the space measured by $t$ and $x$. Thus, we can choose the subsystem $A$ (which are the two charges) to be the length $l$ interval $x\epsilon[-l/2, l/2]$, $y\epsilon[L/2, L/2]$ in the infinitely long total space $-\infty < x < \infty$, see Fig. 2.

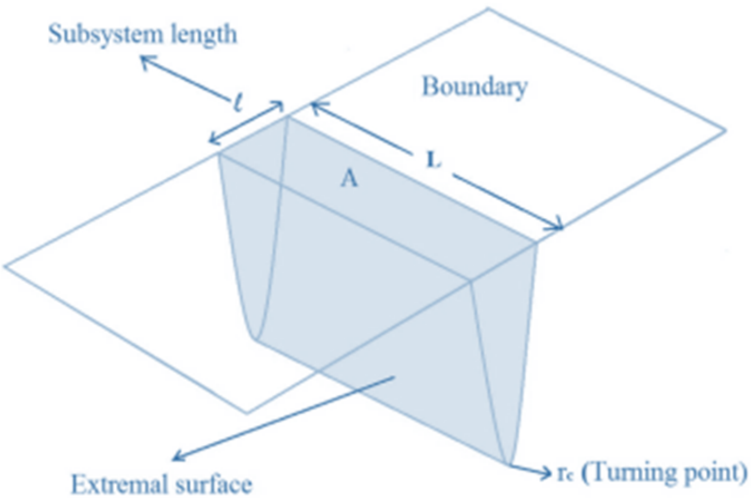


Figure 2: The figure show the schematic of a extremal surface where $l$ is the length of the subsystem $A$, which is anchored on the subsystem living on the boundary.

Now, we will provide the entanglement entropy for the Horndeski gravity following the steps by steps of [28, 29] and motivated by recent studies of [21], we have to the action (7) that the action with boundary counterterms is given by:

$$I_E = I_{bulk} = -\int \sqrt{g} d^4x \mathcal{L} - 2\kappa \int d^3x \sqrt{\bar{\gamma}} \mathcal{L}_b + 2\kappa \int d^3x \sqrt{\bar{\gamma}} \mathcal{L}_{ct}, \tag{22}$$

$$\mathcal{L} = \kappa(R - 2\Lambda) + \frac{\gamma}{2} G_{\mu\nu} \nabla^{\mu}\phi \nabla^{\nu}\phi \tag{23}$$

$$\mathcal{L}_b = K^{(\bar{\gamma})} - \Sigma^{(\bar{\gamma})} + \frac{\gamma}{4}\left(\nabla_\mu\phi\nabla_\nu\phi\, n^\mu n^\nu - (\nabla\phi)^2\right)K^{(\bar{\gamma})} + \frac{\gamma}{4}\nabla^\mu\phi\nabla^\nu\phi K^{(\bar{\gamma})}_{\mu\nu} \quad (24)$$

$$\mathcal{L}_{ct} = c_0 + c_1 R + c_2 R^{ij}R_{ij} + c_3 R^2 + b_1(\partial_i\phi\partial^i\phi)^2 \quad (25)$$

$\mathcal{L}_b$ corresponds to the Gibbons-Hawking $\gamma$-dependent terms associated with the Horndeski gravity, where $n^\mu$ is an outward pointing unit normal vector to the boundary, $K^{(\bar{\gamma})} = \bar{\gamma}^{\mu\nu}K^{(\bar{\gamma})}_{\mu\nu}$ is the trace of the extrinsic curvature and $\bar{\gamma}_{\mu\nu}$ is the induced metric on the boundary $r \to \infty$. The Lagragian $\mathcal{L}_{ct}$ is related to the boundary counterterms, they do not affect the bulk dynamics and will be neglect [31]. Thus, the induced metric to (11) is written as

$$ds^2_{ind} = \bar{\gamma}_{\mu\nu}dx^\mu dx^\nu = \mathcal{R}^2\left(r^2 f(r)d\tau^2 + r^2(dx^2+dy^2) + \frac{dr^2}{r^2 f(r)}\right), \quad (26)$$

With the above, we can provide that the Ryu-Takayanagi formula [7, 8] is given by:

$$S_A = \frac{\mathcal{A}}{4G_N}, \quad (27)$$

$$\mathcal{A} = \int ds_{ind}\chi, \quad (28)$$

$$\chi = 1 - 2\gamma(\bar{\gamma}^{\lambda\sigma}\nabla_\lambda\phi\nabla_\sigma\phi). \quad (29)$$

Here this entropy is equivalent to the cases of [27, 28]. For $AdS_4/CFT_3$, we have that the minimum surface is given by the geodesic line in $AdS_4$. That is, we have following the prescription of [10], we have that the area of the surface anchored on the boundary to the subsystem (A), can be expressed as:

$$\mathcal{A} = 2\chi\mathcal{R}Lr_c\int_0^1 \frac{du}{u^2\sqrt{(1-u^4)f(u)}}; \quad f(u) = 1 - \left(\frac{r_h u}{r_c}\right)^3, \quad (30)$$

where $u = r_c/r$. In our setup to holographic entanglement entropy in Horndeski gravity $r_c$ is a constant of integration, which represents the turning point of the extremal surface in the higher dimensional AdS4 bulk spacetime, see Fig. 2. We can see that through the equation (30) the area integral is divergent at the point $u = 1$ and must be regularized introducing an infrared cutoff ($r_b$). From the point of view of the holographic dictionary, we have a relation between the UV cutoff of the boundary field theory ($\epsilon$) to bulk IR cutoff. Such relation is inversely related through the AdS length scale $\mathcal{R}$ and can be establish as, $r_b = \mathcal{R}/\epsilon$. But note that the finite part of entanglement entropy can then be used to study the high and low temperature behavior of the boundary field theory which is dual to black hole as covered in

$$S^{finite}_A = S_A - S^{divergent}_A = \frac{\mathcal{A}^{finite}}{4G_N}\chi. \quad (31)$$

$$\chi = 1 + \frac{12\kappa\gamma(\alpha+\gamma\Lambda)}{\alpha^2\mathcal{R}^2}. \quad (32)$$

We can to obtain the quantity $r_c$ inverting the equation of motion

$$\frac{l}{2} = \frac{1}{r_c}\int_0^1 \frac{u^2 du}{\sqrt{(1-u^4)f(u)}} \quad (33)$$

As we know, the horizon radius ($r_h$) is very small, we have that the black hole remains deep inside the bulk. Thus, far away from the extremal surface, namely, $r_h << r_c$. For this limit, we can performing Taylor expand. In this case the quantity $1/\sqrt{f(u)}$ will be expanded around $r_h/r_c = 0$ as:

$$\frac{1}{\sqrt{f(u)}} = 1 + \frac{1}{2}\left(\frac{r_h u}{r_c}\right)^3 + \mathcal{O}\left[\left(\frac{r_h u}{r_c}\right)^3\right]^4 \tag{34}$$

Now, replacing this expansion in the equation (33), we can wrote

$$\frac{lr_c}{2} = \int_0^1 \frac{u^2 du}{\sqrt{(1-u^4)}} + \frac{1}{2}\left(\frac{r_h}{r_c}\right)^3 \int_0^1 \frac{u^5 du}{\sqrt{(1-u^4)}} \tag{35}$$

$$\frac{lr_c}{2} = \pi\left(\frac{r_h}{r_c}\right)^3 + \frac{2\sqrt{\pi}\Gamma(3/4)}{\Gamma(1/4)} \tag{36}$$

However, solving the equation (36) in terms of $r_h l$, we have

$$r_c = \frac{2\pi\Gamma(3/4)}{l\Gamma(1/4)} + \frac{l^2\Gamma^3(3/4)}{16\sqrt{\pi}\Gamma^3(1/4)} \tag{37}$$

For the extremal area, we can write following the same steps

$$\mathcal{A} = 2\chi\mathcal{R}Lr_c\left(\int_0^1 \frac{u^2 du}{\sqrt{(1-u^4)}} + \frac{1}{2}\left(\frac{r_h}{r_c}\right)^3 \int_0^1 \frac{udu}{\sqrt{(1-u^4)}}\right) \tag{38}$$

Through the equation (38), we have that first integral is same of a pure AdS, which is divergent. For this reason, we need regularize her, introducing an UV cutoff $1/r_b$ and add a counter term $(-2\mathcal{R}Lr_b)$, after this considerations we provide the finite part of the extremal area as

$$\mathcal{A}^{finite} = 2\chi\mathcal{R}Lr_c\left(\int_0^1 \frac{u^2 du}{\sqrt{(1-u^4)}} - 2\mathcal{R}Lr_b + \frac{1}{2}\left(\frac{r_h}{r_c}\right)^3 \int_0^1 \frac{udu}{\sqrt{(1-u^4)}}\right) \tag{39}$$

$$\mathcal{A}^{finite} = \chi\mathcal{R}Lr_c\left(\sqrt{\pi}\frac{\Gamma(-1/4)}{\Gamma(1/4)} + \pi\left(\frac{r_h}{r_c}\right)^3\right) \tag{40}$$

Combining the equation (40) with (37) the entanglement entropy is given by:

$$S_A^{finite} = \frac{\chi\mathcal{R}L}{4lG_N}\left(-\frac{4\pi\Gamma^2(3/4)}{\Gamma^2(1/4)} + \frac{l^3 r_h^3\Gamma^2(1/4)}{8\Gamma^2(3/4)}\right) \tag{41}$$

For case of extremal black holes we consider $r_h^3 = \mathcal{M}_{ext}/4$ where $\mathcal{M}_{ext}$ is the mass of the extremal black hole, we have

$$S_A^{finite} = [S_A^{AdS} + k\mathcal{M}_{ext}l^2 L]\chi, \quad k = \frac{\mathcal{R}L}{32G_N}\frac{\Gamma^2(1/4)}{\Gamma^2(3/4)} \tag{42}$$

$$S_A^{AdS} = -\frac{4\pi\mathcal{R}L}{4lG_N}\frac{\Gamma^2(3/4)}{\Gamma^2(1/4)} \tag{43}$$

The results shown in the equations (42) and (43) are similar to the case of [10], but in our case, we have the presence of Horndeski parameters, where $S_A^{AdS}$ is the entanglement entropy of the subsystem (A), when the bulk theory is pure AdS as described by [44]. On the other hand, if $\gamma$ is large, the Horndeski contribution through the $\chi$-term in the equation (42), this imply that $S_A^{finite}$ becomes large and the storage of information show that, we have more information about the subsystem (A). Furthermore, at the critical point $\alpha = 3\gamma/\mathcal{R}^2$, for more discussion see [25], the entropy in equation (42) reduces to the usual case of [10] where sub-leading correction term in the equation (42), becomes important in defining the first law like relation.

## 3.1 Planar black hole

We address the issue of finding planar black hole solutions in Horndeski gravity. For this, we consider for Einstein-Horndeski gravity the following *ansatz*:

$$ds^2 = -f(r)dt^2 + r^2(dx^2 + dy^2) + \frac{dr^2}{f(r)}. \tag{44}$$

One can show that the equations (8) and (10) are satisfied by the following solution

$$f(r) = \frac{\alpha r^2}{3\gamma} - \frac{r_h}{r}, \tag{45}$$

$$\psi^2(r) = -\frac{2\kappa(\alpha + \gamma\Lambda)}{\alpha\gamma r^2}\frac{1}{f(r)}. \tag{46}$$

Following the previous procedures, we have

$$\frac{l}{2} = \int_0^1 \frac{du}{u\sqrt{(1-u^4)f(u)}}; \quad f(u) = \frac{\alpha r_c^2}{3\gamma u^2} - \frac{r_h u}{r_c}, \tag{47}$$

$$\mathcal{A} = 2\chi L r_c \int_0^1 \frac{du}{u^2\sqrt{(1-u^4)f(u)}}. \tag{48}$$

Again through the expansion of $1/\sqrt{f(u)}$, we have

$$\frac{1}{\sqrt{f(u)}} = \sqrt{\frac{3\gamma}{\alpha}}\frac{u}{r_c} + \sqrt{\left(\frac{3\gamma}{\alpha}\right)^3}\frac{u^4 r_h}{r_c^4} \tag{49}$$

Now, we can write for the equations (47) and (48) as:

$$\frac{l}{2} = \frac{1}{r_c}\sqrt{\frac{3\gamma}{\alpha}}\frac{\sqrt{\pi}\Gamma(5/4)}{\Gamma(3/4)} + \frac{r_h}{4r_c^4}\sqrt{\left(\frac{3\gamma}{\alpha}\right)^3}, \tag{50}$$

$$\mathcal{A}^{finite} \approx \chi L\sqrt{\frac{3\gamma}{\alpha}}\ln\left(\frac{l}{r_b}\right) - \frac{3\sqrt{3\pi}l\chi L}{4}\sqrt{\frac{\gamma}{\alpha}}\frac{\Gamma(5/4)}{\Gamma(1/4)}. \tag{51}$$

Thus, the entanglement entropy to the planar black is

$$S_A^{finite} = \frac{\mathcal{A}^{finite}}{4G_N} = \frac{\chi L}{4G_N}\sqrt{\frac{3\gamma}{\alpha}}\ln\left(\frac{l}{r_b}\right) - \frac{3\sqrt{\pi}l\chi L}{16G_N}\sqrt{\frac{3\gamma}{\alpha}}\frac{\Gamma(5/4)}{\Gamma(1/4)} \tag{52}$$

When we compare the equation (52) with the equation (42), we have that entanglement entropy to the subsystem (A) have a logarithmic term [5, 6] and a sub-leading correction with Horndeski parameters. Besides, if $\gamma$ is small, the $S_A^{finite} \to 0$ for the planar black hole and all information about the subsystem (A) is destroyed, but not at the critical point $\alpha = 3\gamma/\mathcal{R}^2$ [25] where we have behavior like an area law, which is corrected by a logarithmic factor. Such behavior is finding fermionic systems with the presence of a finite Fermi surface [45, 46].

# 4 Entanglement thermodynamics in Horndeski gravity

In this section, we present the "first law of entanglement thermodynamics". For this, we need of the stress-energy tensor of boundary field theory in Horndeski gravity. Through the renormalization procedure [31] the form of stress-energy tensor $T_{\alpha\beta}$ can be write as:

$$T_{\alpha\beta} = -\frac{r^3}{16\pi\mathcal{R}^3 G_N}\left[K^{(\bar{\gamma})}_{\alpha\beta} - \bar{\gamma}_{\alpha\beta}(K^{(\bar{\gamma})} - \Sigma^{(\bar{\gamma})}) + \frac{\gamma}{4}H_{\alpha\beta} - \kappa T^{R}_{\alpha\beta} - \kappa T^{ct}_{\alpha\beta}\right], \tag{53}$$

$$H_{\alpha\beta} = (\nabla_\alpha\phi\nabla_\beta\phi n^\alpha n^\beta - (\nabla\phi)^2)(K^{(\bar{\gamma})}_{\alpha\beta} - \bar{\gamma}_{\alpha\beta}K^{(\bar{\gamma})}) - (\nabla_\alpha\phi\nabla_\beta\phi)\bar{\gamma}_{\alpha\beta}K^{(\bar{\gamma})}. \tag{54}$$

Here $T^{R}_{\alpha\beta}$ and $T^{ct}_{\alpha\beta}$ are possible contribution of extrinsic curvature and counter term, respectively. However, fixing the energy-momentum tensor on the boundary with $T^{R}_{\alpha\beta} = T^{ct}_{\alpha\beta} = 0$, we have

$$T_{\alpha\beta} = -\frac{r^3}{16\pi\mathcal{R}^3 G_N}\left[K^{(\bar{\gamma})}_{\alpha\beta} - \bar{\gamma}_{\alpha\beta}(K^{(\bar{\gamma})} - \Sigma^{(\bar{\gamma})}) + \frac{\gamma}{4}H_{\alpha\beta}\right]. \tag{55}$$

However, in order to obtain the first law of entanglement thermodynamics in Horndeski gravity we need compute the following quantity:

$$\Delta S_A = \frac{\Delta E_A}{T_{en}}, \tag{56}$$

$$\Delta E_A = \int_A dxdy T^{Temp\neq 0}_{tt} - \int_A dxdy T^{Temp=0}_{tt}. \tag{57}$$

But one interesting physical observable is the mass written as:

$$\mathcal{M} = \int_A dxdy T^{Temp\neq 0}_{tt}, \tag{58}$$

with

$$T_{tt} = -\chi\frac{r^3}{16\mathcal{R}^3\pi G_N}\left[K^{(\bar{\gamma})}_{tt} - \bar{\gamma}_{tt}(K^{(\bar{\gamma})} - \Sigma^{(\bar{\gamma})})\right], \tag{59}$$

$$T_{tt} = \chi\frac{r^3}{8\pi\mathcal{R}^4 G_N}. \tag{60}$$

The result presented by the equation (60) is the same of [31] where the equation (60) corresponds to a excited state in the $CFT_3$ [4]. In this way, we can express the first law of entanglement thermodynamics as:

$$\Delta E_A = \frac{l\chi L}{8\pi G_N}(\mathcal{M} - \mathcal{M}_{ext}), \tag{61}$$

$$T_{en} = \frac{\pi}{16l}\frac{\Gamma^2(3/4)}{\Gamma^2(1/4)}. \tag{62}$$

Where $T_{en}$ is the entanglement temperature [4, 10]. Such results are in perfect agreement with [14]. But here, this result and also as shown in [4], it differs from our results, because we consider a canonical ensemble where the ground state of the boundary field theory is dual to the extremal $AdS_4$ black hole in the bulk.

### 4.1 Planar black hole

Now, for the planar black hole, following the step by step as done to the spherical case we compute the increased amount of energy for such black hole. Thus, through the equation (55), we can show that:

$$T_{tt} = \frac{\chi}{16\pi\mathcal{R}G_N}\sqrt{\frac{3\gamma}{\alpha}}\left(\frac{4}{r} - \frac{1}{R}\right). \tag{63}$$

The equation (63), provide that we can must be not need to make any assumptions in the infrared region $r \to \infty$ for this excited state in the $\mathrm{CFT}_3$. Thus, objects such as black branes or stars in the infrared region can be find. The first has a horizon and is a thermal state, but the second does not have any horizon, however, is dual to a zero temperature state [4]. Furthermore, the increased amount of energy in the subsystem (A) can be written as

$$\Delta E_A = \frac{\chi}{16\pi\mathcal{R}G_N}\sqrt{\frac{\alpha}{3\gamma}}\int_A dydr\frac{4}{r} - \frac{\chi}{16\pi\mathcal{R}^2G_N}\sqrt{\frac{\alpha}{3\gamma}}\int_A dxdy, \tag{64}$$

$$\Delta E_A = \frac{\chi L}{4\pi\mathcal{R}G_N}\sqrt{\frac{3\gamma}{\alpha}}\ln\left(\frac{l}{r_b}\right) - \frac{l\chi L}{16\pi\mathcal{R}^2G_N}\sqrt{\frac{3\gamma}{\alpha}}, \tag{65}$$

$$T_{en} = \frac{1}{3\sqrt{\pi}l}\frac{\Gamma(1/4)}{\Gamma(5/4)}. \tag{66}$$

where the entropy in equation (65) to the planar black hole is the area of the boundary in the region $r \to \infty$. In summary, this result is very similar to the equation (61). On the other hand, when we compare the equation (65) with the equation (52), we see that are very similar. This convergence of results can be done through the renormalization procedure, which removes the logarithmic correction.

## 5 Conclusion

We show in four-dimensions that the study of the entanglement entropy in Horndeski gravity for planar and spherically topologies for a strip like region denoted by the subsystem (A), which is Boundary Conformal Field Theory dual to bulk black holes in an $\mathrm{AdS}_4/\mathrm{CFT}_3$ scenario, provided interesting results, respectively. For these two topologies, we find two interesting aspects of the holographic entanglement entropy. The first behavior of the entanglement entropy with the Horndeski parameters of the AdS black hole can be becomes pure AdS, if kinematic coupling $\gamma$ is very small, that is, $\gamma \to 0$. Thus, such results show that there is similar behavior of study in the field of AdS/CFT correspondence within Horndeski gravity, when if explore the black hole thermodynamics of the black holes, as for example in the studies of [21, 27]. For $\gamma = 0$ we recover the usual entropy of the AdS space.

The second behavior for the entanglement entropy is very interesting, because the values of parameters of Horndeski gravity impose limitations in the storage of information to the planar black holes, if the entanglement entropy of him becomes null, that is, $S_A^{finite} \to 0$ when $\gamma \to 0$ the storage information is completely destroyed, but in the critical points [25] where $\alpha = 3\gamma/\mathcal{R}^2$, we have that the entanglement entropy's described by equation (52) preserve the storage information of the subsystem (A), because are not constraint by the parameters at this point.

Finally, for the first law of entanglement thermodynamics in Horndeski gravity, we show that for spherical case, it has an extremal black holes solution. The first topology of extremal black hole the large implies a large horizon radius. For the second topology, we have that the increasing of energy $\Delta E_A$ is constraint by the Horndeski parameters. This fact, agree with the Ryu-Takayanagi formula as shown in (52). This first law of entanglement thermodynamics in Horndeski gravity in fact agrees with the result of [21, 27, 28, 29].

## Acknowledgment

Fabiano F. Santos would like to thank CNPq and CAPES for partial financial support. I would like to thank Mohammad Hassan Vahidinia for valuable comments and discussions.

## References

[1] T. Takayanagi, "Entanglement Entropy from a Holographic Viewpoint", Class. Quant. Grav. **29** (2012), 153001, [arXiv:1204.2450 [gr-qc]].

[2] A. Bhattacharya, K. T. Grosvenor and S. Roy, "Entanglement Entropy and Subregion Complexity in Thermal Perturbations around Pure-AdS Spacetime", Phys. Rev. D **100**, no.12, 126004 (2019), arXiv:1905.02220 [hep-th]].

[3] A. Bhattacharya and S. Roy, "Holographic entanglement entropy and entanglement thermodynamics of 'black' non-susy D3 brane", Phys. Lett. B **781**, 232-237 (2018), [arXiv:1712.03740 [hep-th]].

[4] J. Bhattacharya, M. Nozaki, T. Takayanagi and T. Ugajin, "Thermodynamical Property of Entanglement Entropy for Excited States", Phys. Rev. Lett. **110**, no.9, 091602 (2013), [arXiv:1212.1164 [hep-th]].

[5] P. Calabrese and J. L. Cardy, "Entanglement entropy and quantum field theory", J. Stat. Mech. **0406** (2004), P06002, [arXiv:hep-th/0405152 [hep-th]].

[6] C. Holzhey, F. Larsen and F. Wilczek, "Geometric and renormalized entropy in conformal field theory", Nucl. Phys. B **424**, 443-467 (1994), [arXiv:hep-th/9403108 [hep-th]].

[7] S. Ryu and T. Takayanagi, "Aspects of Holographic Entanglement Entropy", JHEP **08** (2006), 045, [arXiv:hep-th/0605073 [hep-th]].

[8] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT", Phys. Rev. Lett. **96** (2006), 181602, [arXiv:hep-th/0603001 [hep-th]].

[9] L. Susskind and J. Uglum, "Black hole entropy in canonical quantum gravity and superstring theory", Phys. Rev. D **50** (1994), 2700-2711, [arXiv:hep-th/9401070 [hep-th]].

[10] P. Chaturvedi, V. Malvimat and G. Sengupta, "Entanglement thermodynamics for charged black holes", Phys. Rev. D **94**, no.6, 066004 (2016), [arXiv:1601.00303 [hep-th]].

[11] E. Tonni, "Holographic entanglement entropy: near horizon geometry and disconnected regions", JHEP **05** (2011), 004, [arXiv:1011.0166 [hep-th]].

[12] S. A. H. Mansoori, B. Mirza, M. D. Darareh and S. Janbaz, "Entanglement Thermodynamics of the Generalized Charged BTZ Black Hole", Int. J. Mod. Phys. A **31** (2016) no.12, 1650067, [arXiv:1512.00096 [gr-qc]].

[13] P. Caputa, V. Jejjala and H. Soltanpanahi, "Entanglement entropy of extremal BTZ black holes", Phys. Rev. D **89** (2014) no.4, 046006, [arXiv:1309.7852 [hep-th]].

[14] D. D. Blanco, H. Casini, L. Y. Hung and R. C. Myers, "Relative Entropy and Holography", JHEP **08** (2013), 060, [arXiv:1305.3182 [hep-th]].

[15] L. Susskind, "Entanglement and Chaos in De Sitter Holography: An SYK Example", [arXiv:2109.14104 [hep-th]].

[16] B. S. Kay, "Entanglement entropy and algebraic holography", [arXiv:1605.07872 [hep-th]].

[17] C. Park, "Holographic entanglement entropy in the nonconformal medium", Phys. Rev. D **91** (2015) no.12, 126003, [arXiv:1501.02908 [hep-th]].

[18] S. He, J. R. Sun and H. Q. Zhang, "On Holographic Entanglement Entropy with Second Order Excitations", Nucl. Phys. B **928** (2018), 160-181, [arXiv:1411.6213 [hep-th]].

[19] F. F. Santos, "Aplicações do Setor John da Gravidade de Horndeski nos Cenários de Brana Negra e Relação de viscosidade/entropia, Mundo Brana e Cosmologia (In Portuguese)", [arXiv:2006.06550 [hep-th]].

[20] F. A. Brito and F. F. Santos, "Black brane in asymptotically Lifshitz spacetime and viscosity/entropy ratios in Horndeski gravity", EPL **129**, no.5, 50003 (2020), [arXiv:1901.06770 [hep-th]].

[21] F. F. Santos, E. F. Capossoli and H. Boschi-Filho, "AdS/BCFT correspondence and BTZ black hole thermodynamics within Horndeski gravity", Phys. Rev. D **104**, no.6, 066014 (2021), [arXiv:2105.03802 [hep-th]].

[22] W. J. Jiang, H. S. Liu, H. Lu and C. N. Pope, "DC Conductivities with Momentum Dissipation in Horndeski Theories", JHEP **1707**, 084 (2017), [arXiv:1703.00922 [hep-th]].

[23] M. Baggioli and W. J. Li, "Diffusivities bounds and chaos in holographic Horndeski theories", JHEP **1707**, 055 (2017), [arXiv:1705.01766 [hep-th]].

[24] H. S. Liu, "Violation of Thermal Conductivity Bound in Horndeski Theory", Phys. Rev. D **98**, no. 6, 061902 (2018), [arXiv:1804.06502 [hep-th]].

[25] Y. Z. Li and H. Lu, "$a$-theorem for Horndeski gravity at the critical point", Phys. Rev. D **97**, no. 12, 126008 (2018), [arXiv:1803.08088 [hep-th]].

[26] Y. Z. Li, H. Lu and H. Y. Zhang, "Scale Invariance vs. Conformal Invariance: Holographic Two-Point Functions in Horndeski Gravity", Eur.Phys.J.C **79**, 592 (2019), arXiv:1812.05123 [hep-th].

[27] X. H. Feng, H. S. Liu, H. Lü and C. N. Pope, "Black Hole Entropy and Viscosity Bound in Horndeski Gravity", JHEP **1511**, 176 (2015), [arXiv:1509.07142 [hep-th]].

[28] E. Caceres, R. Mohan and P. H. Nguyen, "On holographic entanglement entropy of Horndeski black holes", JHEP **10**, 145 (2017), [arXiv:1707.06322 [hep-th]].

[29] K. Hajian, S. Liberati, M. M. Sheikh-Jabbari and M. H. Vahidinia, "On Black Hole Temperature in Horndeski Gravity", Phys. Lett. B **812**, 136002 (2021), [arXiv:2005.12985 [gr-qc]].

[30] M. R. Mohammadi Mozaffar, A. Mollabashi, M. M. Sheikh-Jabbari and M. H. Vahidinia, "Holographic Entanglement Entropy, Field Redefinition Invariance and Higher Derivative Gravity Theories", Phys. Rev. D **94**, no.4, 046002 (2016), [arXiv:1603.05713 [hep-th]].

[31] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity", Commun. Math. Phys. **208**, 413 (1999), [hep-th/9902121].

[32] G. W. Horndeski, "Second-order scalar-tensor field equations in a four-dimensional space", Int. J. Theor. Phys. **10**, 363 (1974).

[33] F. F. Santos, "Rotating black hole with a probe string in Horndeski Gravity", Eur. Phys. J. Plus **135**, no.10, 810 (2020), [arXiv:2005.10983 [hep-th]].

[34] A. Cisterna and C. Erices, "Asymptotically locally AdS and flat black holes in the presence of an electric field in the Horndeski scenario", Phys. Rev. D **89**, 084038 (2014), [arXiv:gr-qc/1401.4479].

[35] M. Bravo-Gaete and M. Hassaine, "Thermodynamics of a BTZ black hole solution with an Horndeski source", Phys. Rev. D **90**, no.2, 024008 (2014), [arXiv:1405.4935 [hep-th]].

[36] A. Anabalon, A. Cisterna and J. Oliva, "Asymptotically locally AdS and flat black holes in Horndeski theory", Phys. Rev. D **89**, 084050 (2014), [arXiv:gr-qc/1312.3597 [gr-qc]].

[37] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, "General second order scalar-tensor theory, self tuning, and the Fab Four", Phys. Rev. Lett. **108**, 051101 (2012), [arXiv:1106.2000 [hep-th]].

[38] C. Charmousis, E. J. Copeland, A. Padilla and P. M. Saffin, "Self-tuning and the derivation of a class of scalar-tensor theories", Phys. Rev. D **85**, 104040 (2012), [arXiv:1112.4866 [hep-th]].

[39] A. A. Starobinsky, S. V. Sushkov and M. S. Volkov, "The screening Horndeski cosmologies", JCAP **1606**, 007 (2016), [arXiv:1604.06085 [hep-th]].

[40] J. P. Bruneton, M. Rinaldi, A. Kanfon, A. Hees, S. Schlogel and A. Fuzfa, "Fab Four: When John and George play gravitation and cosmology", Adv. Astron. **2012**, 430694 (2012), [arXiv:1203.4446 [gr-qc]].

[41] L. Hui and A. Nicolis, "No-Hair Theorem for the Galileon", Phys. Rev. Lett. **110**, 241104 (2013), [arXiv:1202.1296 [hep-th]].

[42] M. Bravo-Gaete and M. Hassaine, "Lifshitz black holes with a time-dependent scalar field in a Horndeski theory", Phys. Rev. D **89**, 104028 (2014), [arXiv:1312.7736 [hep-th]].

[43] E. Babichev and C. Charmousis, "Dressing a black hole with a time-dependent Galileon", JHEP **1408**, 106 (2014), [arXiv:1312.3204 [gr-qc]].

[44] W. Fischler and S. Kundu, "Strongly Coupled Gauge Theories: High and Low Temperature Behavior of Non-local Observables", JHEP **05**, 098 (2013), [arXiv:1212.2643 [hep-th]].

[45] D. Gioev and I. Klich, "Entanglement Entropy of Fermions in Any Dimension and the Widom Conjecture", Phys. Rev. Lett. **96**, 100503 (2006), [arXiv:quant-ph/0504151 [quant-ph]].

[46] M. M. Wolf, "Violation of the entropic area law for Fermions", Phys. Rev. Lett. **96**, 010404 (2006), [arXiv:quant-ph/0503219 [quant-ph]].