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GUP-Deformed Thermodynamic Behavior of Cylindrical Black Strings in Four Dimensions

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Abstract. We investigate the thermodynamic properties of four-dimensional cylindrically symmetric black strings embedded in an Anti-de Sitter (AdS) spacetime. By incorporating the Generalized Uncertainty Principle (GUP), we introduce minimal length corrections to the thermodynamic framework, thereby capturing quantum gravity effects in the near-horizon regime. We demonstrate that these GUP-induced modifications lead to logarithmic corrections in the entropy-area relation, particularly for small horizon radii, while preserving thermodynamic consistency with the first law of thermodynamics. Through a detailed analysis of the Helmholtz and Gibbs free energies, we observe that the equation of state is significantly deformed by quantum corrections, driving instabilities in black strings with sufficiently small horizon radii. The study of specific heat capacity further confirms a phase transition from stable to unstable configurations, highlighting the role of the minimal length scale on the thermal stability of black string systems.

Keywords: Black Strings; Thermal Fluctuations; Logarithmic Entropy; Black Hole Thermodynamics.

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1 Background and Research Motivation

Black-hole radiation and evaporation [1,2] reveal a deep interplay between general relativity, thermodynamics and quantum theory. Although black holes are not directly observable as such, theoretical consistency and indirect observations strongly support their existence. The result that a black hole's entropy is proportional to the area of its event horizon [3,4] is perhaps the most important result of black-hole thermodynamics. For large black holes, this is an exact result, but statistical fluctuations near thermal equilibrium produce significant corrections for small black holes [5,6]. Such corrections typically appear as leading-order logarithmic terms in the entropy expansion [7–15].

Recent work has explored how small black holes respond thermodynamically to such statistical fluctuations [16–23]. Four parameters—mass, angular momentum, electric charge, and the cosmological constant—are most commonly employed to characterize black hole solutions. Such solutions may be classified into spherically or cylindrically symmetric configurations and often possess axial symmetry. A well-known spherically symmetric solution is the Schwarzschild metric. A less explored class of solutions is their cylindrical analogues, such as black strings [24]. Cylindrically symmetric black strings and black holes exhibit comparable thermodynamic features in four-dimensional spacetimes, despite their distinct geometries.

Spherically symmetric black holes were primarily the target of the initial research. The hoop conjecture posits that if a hoop of circumference 2 pi r_s can be passed around the mass in every direction, a black hole forms. The original formulation of the hoop conjecture assumes a vanishing cosmological constant; extensions to (A)dS backgrounds remain under discussion. A negative cosmological constant supports cylindrically symmetric solutions, including black strings. Recently, black strings with negative cosmological constants have been constructed within four-dimensional Einstein gravity [25]. Nevertheless, their thermodynamic properties, especially in the regime of small horizon radii where thermal fluctuations dominate, remain insufficiently explored.

This gap in the literature serves as the primary motivation for the current research. We study black strings in AdS spacetimes and analyse their thermodynamic behaviour. Entropy is largely determined by the surface area of the event horizon, and statistical fluctuations are small for large black strings. Such fluctuations are no longer negligible for small black strings. However, classical expressions for the entropy must be modified. We compute the corrected entropy and examine its impact on the thermodynamic properties of the system. Specifically, we examine the effect of temperature fluctuations on the thermodynamic equations of state for the black string. Our analysis shows that the thermodynamic behaviour is substantially modified for black strings with small horizon area once entropy corrections are included.

The structure of this paper is as follows: In Section 2, we present the geometric setup of the black string solution and its associated thermodynamic properties. Section 3 explores the implications of the Generalized Uncertainty Principle (GUP) on the thermodynamic framework. In Section 4, we analyze the behavior of entropy under GUP corrections. Section 5 extends the discussion to the non-equilibrium thermodynamic regime of black strings. The criteria for thermodynamic stability and the concept of compressibility are examined in Section 6. Finally, Section 7 summarizes our findings and outlines possible directions for future research.

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2 Black String Geometry and Thermodynamic Properties

In the presence of a cosmological constant and with $G = 1$, the Einstein-Hilbert action reads [26]:

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R + 6\eta^2), \tag{2.1}$$

where R is the Ricci scalar.

The metric tensor is $g_{\mu\nu}$, and the parameter η is defined by $\eta^2 = -\frac{\Lambda}{3} > 0$ with $\Lambda < 0$. Variation of the action in Eq. (2.1) yields the following field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 3\eta^2 g_{\mu\nu}, \tag{2.2}$$

where $R_{\mu\nu}$ is the Ricci tensor.

The metric solution to the field equations in a cylindrically symmetric and static space-time with coordinates (t, r, ϕ, z) is provided by [24,26]:

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2d\phi^2 + \alpha^2r^2dz^2, \tag{2.3}$$

where $-\infty < t, z < +\infty$, $0 \leq r < +\infty$, and $0 \leq \phi \leq 2\pi$ with these coordinate ranges. The metric function is given by [26]:

$$f(r) = g^{-1}(r) = \left(\alpha^2r^2 - \frac{4M}{\alpha r}\right), \tag{2.4}$$

where the ADM (Arnowitt-Deser-Misner) mass density of the black string is represented by the integration constant M. There is a curvature singularity at $r = 0$, but the present analysis does not focus on this singularity.

The equation $f(r_+) = 0$ can be solved to find the event horizon. The event horizon for positive M is at:

$$r_+ = \frac{1}{\alpha}(4M)^{\frac{1}{3}},$$

with an naked singularity' to 'with a naked singularity r. For negative M, the horizon disappears and a naked singularity remains.

The thermodynamics of black string solutions can be examined using the Euclidean method [27,28]. The periodicity condition for Euclidean time and the Hawking temperature are obtained by requiring that there be no conical singularity in Euclidean spacetime:

$$T_H = \frac{3\alpha^2 r_+}{4\pi}. \tag{2.5}$$

As an alternative, the Hawking temperature derived from the surface gravity is:

$$T_H = \left(\frac{3\alpha}{2\pi}\right)\left(\frac{M}{2}\right)^{1/3},$$

which, in contrast to the Schwarzschild black hole, scales as $M^{1/3}$. This demonstrates how the geometry affects the thermodynamics of the black string. Treating the black string as a thermodynamic system, the first law reads [29,30]:

$$dM = T_H dS_0, \tag{2.6}$$

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where the equilibrium entropy is denoted by S_0. Although PV terms appear in the extended thermodynamic phase space [29], they are not included in the present formulation [29,30].

By using Eqs. (2.5) and (2.6), the entropy can be computed as follows:

$$S_0 = \int \frac{dM}{T_H} = \frac{\pi\alpha r_+^2}{2}. \tag{2.7}$$

Substituting r_+ from above, the entropy reads:

$$S_0 = \frac{\pi}{2\alpha}(4M)^{2/3},$$

which deviates from the standard black hole entropy and scales as $M^{2/3}$. The area law states that $S_0 = \sigma/4$, where σ is the event horizon area, which can be written as follows:

$$\sigma = 2\pi\alpha r_+^2 = \frac{\pi}{2\alpha}\left(4M\right)^{\frac{2}{3}}.$$

The thermodynamic volume V is thus determined by:

$$V = \int \sigma dr_+ = \frac{2}{3}\pi\alpha r_+^3. \tag{2.8}$$

The analysis of how minor statistical fluctuations alter the equilibrium thermodynamic behavior is made possible by these expressions; this effect is particularly significant for small black strings near the Planck regime.

3 Impact of the Generalized Uncertainty Principle

Various quantum gravity frameworks, such as loop quantum gravity and string theory, propose modifications to the dispersion relation governing photon propagation in vacuum. These deviations are frequently modeled using modified dispersion relations (MDRs), which may yield observable implications across cosmological distances [31]. In models introducing a minimal length scale—acting as an effective ultraviolet (UV) cut-off—phenomenological studies of quantum gravity often employ Generalized Uncertainty Principles (GUPs) and Modified Dispersion Relations (MDRs). Recent work has elucidated the complementarity between these two frameworks [32]. Conceptually, the notion of a minimal length can be implemented via two primary methods: either by adopting a discretized spacetime structure, a concept tracing back to early developments in quantum field theory [33], or by modifying the canonical Heisenberg algebra [34,35], a process that naturally yields generalized uncertainty principles.

In this section, we adopt the latter approach. As demonstrated in non-anticommutative superspace field theory [36,37] and canonical noncommutative field theory using coherent state formalism [39], the Feynman propagator acquires an exponential UV cut-off of the form $\exp\left(-\eta p^2\right)$, where η characterizes the smallest length scale. This technique has been successfully applied in the analysis of black hole evaporation [36]. To express the UV finiteness of the Feynman propagator at the quantum level, a nonlinear relation between the wave vector and the physical momentum, $p = f(k)$ [32], is utilized. For this function to be invertible, two conditions must be satisfied:

1. The conventional dispersion relation must be recovered in the infrared (low-energy) limit.

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2. The wave vector must asymptotically approach a maximum (cut-off) value in the high-energy regime.

These require modifying the momentum-space measure from $d^n p$ to $d^n p \prod_i \frac{\partial k_i}{\partial p_j}$. To simplify the analysis, we consider an isotropic system with a single spatial dimension. Following [39] and defining $\eta = \frac{\alpha^2 L_{Pl}^2}{\hbar^2}$, we obtain:

$$\frac{\partial p}{\partial k} = \hbar \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2} p^2\right), \tag{3.1}$$

Here, α represents a dimensionless constant of order unity.

By integrating Eq. (3.1), the corresponding dispersion relation is determined to be:

$$k\left(p\right) = \frac{\sqrt{\pi}}{2\alpha L_{Pl}} \mathrm{erf}\left(\frac{\alpha L_{Pl}}{\hbar} p\right), \tag{3.2}$$

where the minimal Compton wavelength is given by:

$$\lambda_0 = 4\sqrt{\pi}\alpha L_{Pl}. \tag{3.3}$$

We now demonstrate that a specific realization of the position and momentum operators naturally produces these results:

$$X = i\hbar \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2} P^2\right) \partial_p \qquad P = p. \tag{3.4}$$

Significant deviations from the standard Heisenberg algebra emerge as one approaches the Planck regime, i.e. $p \sim M_{Pl}$ and $\delta x \sim L_{Pl}$.

To ensure the hermiticity of the position operator within this formalism, we present a corrected completeness relation:

$$\int dp e^{-\frac{\alpha^2 L_{Pl}^2}{\hbar^2} p^2} |p\rangle\langle p| = 1, \tag{3.5}$$

along with a revised scalar product:

$$\langle p|\, p'\rangle = e^{\frac{\alpha^2 L_{Pl}^2}{\hbar^2} p^2} \delta\left(p - p'\right). \tag{3.6}$$

By applying Eq. (3.5), one can identify the Gaussian damping factor that characterizes the Feynman propagator [36,38]. The operator algebra defined in Eq. (3.4) subsequently yields the generalized commutation relation and the associated generalized uncertainty principle (GUP*):

$$[X, P] = i\hbar \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2} P^2\right), \quad \left(\delta X\right)\left(\delta P\right) \geq \frac{\hbar}{2} \left\langle \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2} P^2\right)\right\rangle. \tag{3.7}$$

Focusing on the saturated GUP*, we examine the quantum consequences of this deformed algebra to derive an expression for the momentum uncertainty, (δP). Utilizing the inequality $\left\langle P^{2n}\right\rangle \geq \left\langle P^2\right\rangle^n$ and the statistical relation $\left(\delta P\right)^2 = \left\langle P^2\right\rangle - \left\langle P\right\rangle^2$, we arrive at:

$$\left(\delta X\right)\left(\delta P\right) = \frac{\hbar}{2} \exp\left(\frac{\alpha^2 L_{Pl}^2}{\hbar^2} \left(\left(\delta P\right)^2 + \left\langle P\right\rangle^2\right)\right). \tag{3.8}$$

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Squaring and rearranging the expression above leads to the defining equation for the Lambert W function:

$$W\left(u\right)e^{W\left(u\right)} = u, \tag{3.9}$$

where $W(u) = -2\frac{\alpha^2 L_{Pl}^2}{\hbar^2}\left(\delta P\right)^2$ and $u = -\frac{\alpha^2 L_{Pl}^2}{2(\delta X)^2} e^{-2\frac{\alpha^2 L_{Pl}^2}{\hbar^2}\langle P\rangle^2}$. Equation (3.9) defines the multivalued Lambert W function, where the branches are characterized by integers $k = 0, \pm1, \pm2, \ldots$. For the range $-\frac{1}{e} \leq u \leq 0$, there exist two real-valued solutions: $W_0(u)$ and $W_{-1}(u)$. For $u \geq 0$, only $W_0(u)$ is physically valid, while no real solutions exist for $u < -\frac{1}{e}$.

Consequently, the uncertainty in momentum, (δP), can be expressed as:

$$\left(\delta P\right) = \frac{\hbar e^{\frac{\alpha^2 L_{Pl}^2}{\hbar^2}\langle P\rangle^2}}{2\left(\delta X\right)} \exp\left(-\frac{1}{2}W\left(-\frac{\alpha^2 L_{Pl}^2 e^{\frac{2\alpha^2 L_{Pl}^2}{\hbar^2}\langle P\rangle^2}}{2\left(\delta X\right)^2}\right)\right). \tag{3.10}$$

To ensure the expression remains real, the argument of the Lambert function must satisfy the following condition:

$$\frac{\alpha^2 L_{Pl}^2 e^{\frac{2\alpha^2 L_{Pl}^2}{\hbar^2}\langle P\rangle^2}}{2\left(\delta X\right)^2} \leqslant \frac{1}{e}, \tag{3.11}$$

which establishes a lower bound for the position uncertainty:

$$\left(\delta X\right)_{\min} = \sqrt{\frac{e}{2}}\alpha L_{Pl} e^{\frac{\alpha^2 L_{Pl}^2}{\hbar^2}\langle P\rangle^2}. \tag{3.12}$$

For physical states where $\langle P\rangle = 0$ and $(\delta P) = \hbar/\left(\sqrt{2}\alpha L_{Pl}\right)$, we find the absolute minimum uncertainty in position:

$$\left(\delta X\right)_0 = \sqrt{\frac{e}{2}}\alpha L_{Pl}. \tag{3.13}$$

This result allows us to rewrite the momentum uncertainty in terms of the minimum length:

$$\left(\delta P\right) = \frac{\hbar}{2\left(\delta X\right)} \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{(\delta X)_0}{(\delta X)}\right)^2\right)\right). \tag{3.14}$$

Finally, the modified uncertainty relation takes the form:

$$\delta X \delta P = \frac{\hbar}{2} \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{(\delta X)_0}{\delta X}\right)^2\right)\right). \tag{3.15}$$

Since the term $-\frac{1}{e}\frac{(\delta X)_0}{(\delta X)} < 1$ acts as a small expansion parameter, a systematic perturbative series expansion of the generalized uncertainty principle (GUP*) becomes feasible in the perturbative (sub-Planckian) regime.

4 Entropy Behavior of Black Strings under GUP Corrections

Following the GUP framework established in the previous section, we now explore its implications for the microscopic (microcanonical) formulation of entropy for black strings. As noted, the inclusion of a fundamental Planck length becomes physically significant in strong

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gravitational regimes near black hole horizons. Our objective is to quantify how these quantum gravitational corrections modify the standard thermodynamic relations. According to Bekenstein's seminal work, classical black hole entropy is proportional to the area of the event horizon. Furthermore, when a particle is captured by the black hole, the horizon surface area must increase by a minimum specific increment. Following the absorption of a classical particle with energy E and spatial size R, this minimum area increase is typically given by:

$$(\Delta A)_0 \simeq 4L_{Pl}^2(\ln 2)(ER). \tag{4.1}$$

At the quantum level, constraints on the particle's energy and spatial extent arise, which can be expressed in terms of position and momentum uncertainties:

$$R \simeq 2\delta X \quad \text{and} \quad E \simeq \delta P. \tag{4.2}$$

By considering the near-horizon geometry and applying the *GUP**, we refine the formula for the minimum change in the horizon area:

$$(\Delta A)_0 \approx 4L_{Pl}^2(\ln 2)(2\delta X\delta P). \tag{4.3}$$

Utilizing the specific functional form involving the Lambert W-function and the deformed commutation relations induced by *GUP*, this becomes:

$$(\Delta A)_0 \approx 4L_{Pl}^2(\ln 2)\hbar \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{(\delta X)_0}{\delta X}\right)^2\right)\right). \tag{4.4}$$

By replacing $A = 4\pi(\delta X)^2$ and $A_0 = 4\pi(\delta X)_0^2$, the expression above can be rewritten as:

$$(\Delta A)_0 \approx 4L_{Pl}^2(\ln 2)\left[\exp\left(-\frac{1}{2}W\left(-\frac{A_0}{eA}\right)\right)\right]. \tag{4.5}$$

Assuming the minimum entropy increase is $(\Delta S)_0 = \ln 2$, the differential entropy-area relation is given by:

$$\frac{dS}{dA} \approx \frac{1}{4L_{Pl}^2}\left[\exp\left(-\frac{1}{2}W\left(-\frac{A_0}{eA}\right)\right)\right]. \tag{4.6}$$

To determine the total entropy, we integrate from the minimum area A_0 to a generic horizon area A:

$$S \simeq \frac{1}{4L_{Pl}^2}\int_{A_0}^{A} \exp\left[\frac{1}{2}W\left(-\frac{A_0}{eA'}\right)\right] dA. \tag{4.7}$$

Using the identity $e^{W(x)/2} = \sqrt{x/W(x)}$, the entropy integral can be reformulated as:

$$S = \frac{A_0}{4eL_{Pl}^2}\mathbf{PV}\int_{-\frac{1}{e}}^{-\frac{A_0}{eA}} y^{-3/2}\left[W(y)\right]^{-1/2} dy. \tag{4.8}$$

where PV denotes the Cauchy Principal Value of the integral. By switching variables using $y = -\frac{A_0}{eA}$ and calculating the integral, we obtain a direct expression for the corrected entropy:

$$S = \frac{A_0}{8eL_{Pl}^2}\left[\mathrm{Ei}\left(-\frac{1}{2}W\left(-\frac{A_0}{eA}\right)\right) - 2\left(-\frac{A_0}{eA}W\left(-\frac{A_0}{eA}\right)\right)^{-1/2} - 2\sqrt{e} - \mathrm{Ei}\left(\frac{1}{2}\right)\right]. \tag{4.9}$$

Here, $\mathrm{Ei}(x)$ represents the exponential integral function.

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Expanding the entropy in a power series around $\left(\frac{A_0}{eA}\right)$ yields:

$$S = \frac{A}{4L_{Pl}^2} - \frac{A_0}{8eL_{Pl}^2}\ln\left(\frac{A}{A_0}\right) + \frac{3\pi\eta^2}{16e}\left(\frac{A_0}{A}\right) + \frac{25\pi\eta^2}{192e^2}\left(\frac{A_0}{A}\right)^2 + \cdots + C. \tag{4.10}$$

In the large area limit ($A \gg A_0$), the higher-order terms proportional to powers of (A_0/A) are negligible. Truncating these sub-leading terms simplifies the expression to:

$$S \approx \frac{A}{4L_{Pl}^2} - \frac{A_0}{8eL_{Pl}^2}\ln\left(\frac{A}{A_0}\right) + C. \tag{4.11}$$

The minimum area A_0 is defined by the minimum position uncertainty $(\delta X)_0 = \sqrt{e/2}\alpha L_{Pl}$, leading to $A_0 = 4\pi(\delta X)_0^2 = 2\pi e\alpha^2 L_{Pl}^2$. Substituting this identity into the logarithmic coefficient gives:

$$-\frac{A_0}{8eL_{Pl}^2} = -\frac{2\pi e\alpha^2 L_{Pl}^2}{8eL_{Pl}^2} = -\frac{\pi\alpha^2}{4}. \tag{4.12}$$

By identifying the GUP deformation parameter α^2 with η^2 and defining the fluctuation parameter $\beta = -\frac{\pi\eta^2}{4}$, we can rewrite the logarithmic term. Absorbing the constant term $\ln(4L_{Pl}^2/A_0)$ into the integration constant C allows the entropy to be expressed in the concise form:

$$S = \frac{A}{4L_{Pl}^2} + \beta\ln\left(\frac{A}{4L_{Pl}^2}\right) + \cdots \tag{4.13}$$

Letting $S_0 = \frac{A}{4L_{Pl}^2}$ represent the leading-order (equilibrium) entropy, the corrected microcanonical entropy becomes:

$$S = S_0 + \beta\ln S_0 + \cdots \tag{4.14}$$

The coefficient β quantifies the entropy correction due to quantum fluctuations. The logarithmic term plays an essential role in determining entropy behavior for small black holes. This confirms that the entropy counts microstates associated with horizon degrees of freedom, serving as a measure of the number of microstates. For the black-string spacetime considered here, the equilibrium entropy reads:

$$S_0 = \frac{\pi\alpha r_+^2}{2}. \tag{4.15}$$

Substituting (4.15) into the corrected microcanonical expression (4.14) yields:

$$S = \frac{\pi\alpha r_+^2}{2} + \beta\ln(2\pi\alpha r_+^2). \tag{4.16}$$

Equation (4.16) enables a quantitative analysis of the quantum corrections as a function of the horizon radius r_+.

The corrected entropy approaches the classical Bekenstein-Hawking value for $\beta \to 0$, as is evident from Fig. 1. The quantum corrections produce significant deviations for small r_+ values. The system is stabilized for negative values of β since it leads to a finite asymptotic entropy. In contrast, a positive value of β leads to a negative asymptotic entropy at small values of r_+, which is physically unacceptable.

Notably, it is possible to reinterpret this finding by applying the Bekenstein-Hawking equation $S_0 = \frac{A}{4G}$. The appearance of negative entropy suggests a breakdown of the semi-classical approximation. Theoretical interpretations in this regime often invoke a sign reversal of G, mimicking repulsive gravitational forces. Perhaps derived from the relativistic concept of antiparticles as proposed in Dirac's equation, this theoretical basis is in agreement with cosmological models that attribute cosmic expansion to repulsive gravitational forces.

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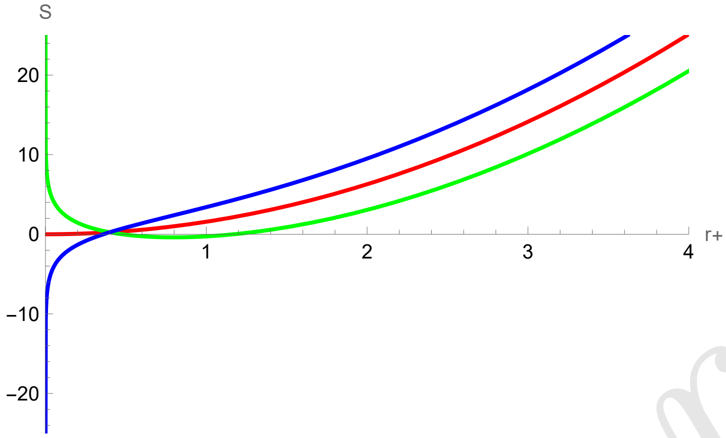

Figure 1: Entropy S versus black-string horizon radius r_+. Curves correspond to various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

5 Black String Thermodynamics in the Non-Equilibrium Regime

In classical thermodynamics, the first law identifies the internal energy E as the central state variable. However, it is the change in internal energy, and not its absolute value, that is physically meaningful in non-equilibrium thermodynamic regimes considered here.

We use the Hawking temperature from Eq. (2.5) and the corrected entropy from Eq. (4.16) to obtain the quantum-corrected internal energy of the black string via the integral expression:

$$E = \int T_H \, dS. \tag{5.1}$$

which yields:

$$E = \frac{\alpha^2(6\beta r_+ + \alpha\pi r_+^3)}{4\pi}. \tag{5.2}$$

The variation of the internal energy E with respect to the event horizon radius r_+ of the black string is illustrated in Figure 2. As one can observe, when r_+ increases, the internal energy also increases. The internal energy is modified by the existence of thermal fluctuations, which are parametrised by the correction parameter β, particularly for small black strings. Positive (negative) values of β signal a departure from equilibrium behaviour, decreasing (increasing) the internal energy.

The Helmholtz free energy F, a thermodynamic potential that is crucial to understanding the stability and phase structure of black hole systems, is then evaluated, given by:

$$F = -\int S \, dT_H,$$

We utilize the Hawking temperature and corrected entropy formulas to obtain:

$$F = \frac{-\alpha^2 r_+(-12\beta + \alpha\pi r_+^2 + 6\beta\log(2\alpha\pi r_+^2))}{8\pi}. \tag{5.3}$$

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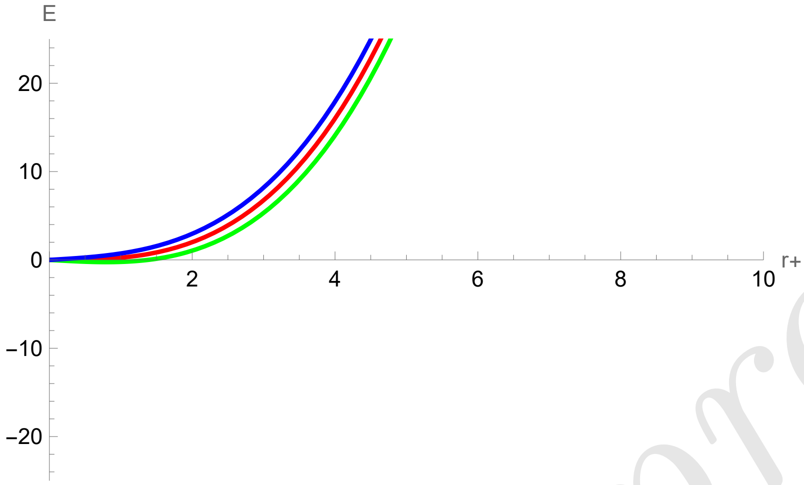

Figure 2: Internal energy E versus black-string horizon radius r_+. Curves correspond to various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

The response of the Helmholtz free energy to the horizon radius is plotted in Figure 3. As r_+ increases, the Helmholtz free energy F decreases monotonically. Positive values of β enhance the rate of decay of F and hence improve the thermodynamic stability, whereas the equilibrium free energy profile is regained for $\beta = 0$. Negative values of β, however, retard this decay and can temporarily increase F for small black strings, which delays the approach to thermodynamic equilibrium.

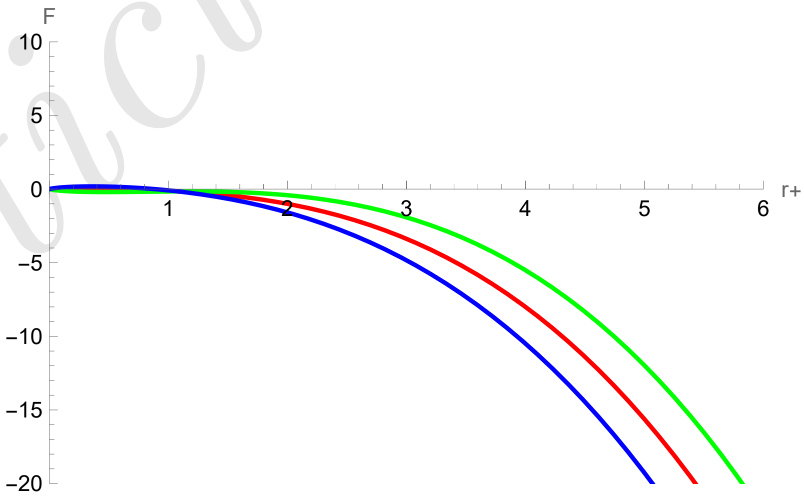

Figure 3: Helmholtz free energy F versus the event horizon radius r_+. Curves are for various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

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We use the relation to find the thermodynamic pressure of the black string:

$$P = -\left(\frac{\partial F}{\partial V}\right), \tag{5.4}$$

with the volume expression of Eq. (2.8) and the free energy of Eq. (5.3). This gives:

$$P = \frac{3\alpha(\alpha\pi r_+^2 + 2\beta\log(2\alpha\pi r_+^2))}{16\pi^2 r_+^2}. \tag{5.5}$$

The pressure as a function of r_+ is illustrated in Figure 4. For large horizon radii, the pressure is essentially independent of β; for small r_+ it becomes strongly β-dependent for small r_+ it becomes strongly β-dependent. For negative β, the pressure changes sign and diverges as $r_+ \to 0$. Close to the end-point of black string evaporation, this divergence signals the onset of strong tidal forces reflecting the divergence of curvature invariants near the singularity.

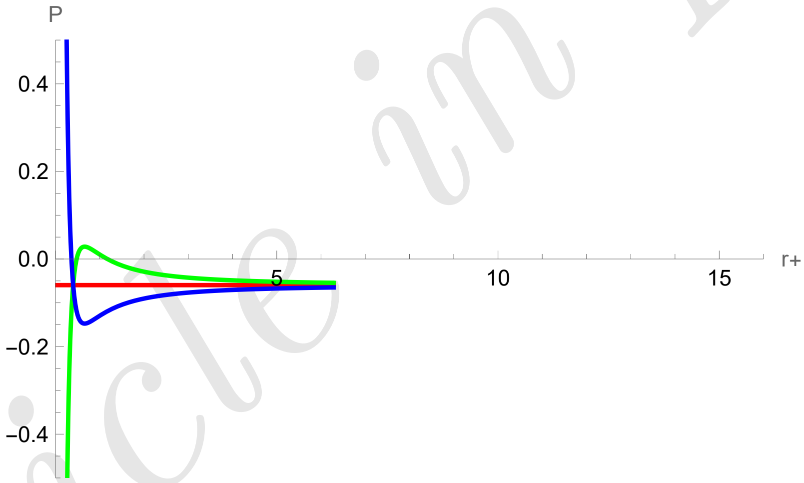

Figure 4: Thermodynamic pressure P versus black-string horizon radius r_+. Curves correspond to various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

The maximum amount of work that can be extracted from the system at constant temperature and pressure is measured by the Gibbs free energy, which is calculated as follows:

$$G = \frac{-\alpha^2 r_+(-6\beta + \alpha\pi r_+^2 + 4\beta\log(2\pi\alpha r_+^2))}{4\pi}. \tag{5.6}$$

Fig. 5 shows how G tends to decrease with larger r_+. A positive correction parameter accelerates the decrease of G, indicating a faster approach to thermodynamic stability. Small black strings, however, can lead to instability if this tendency is temporarily reversed. Thermodynamic behavior deviates significantly from the equilibrium case for negative β.

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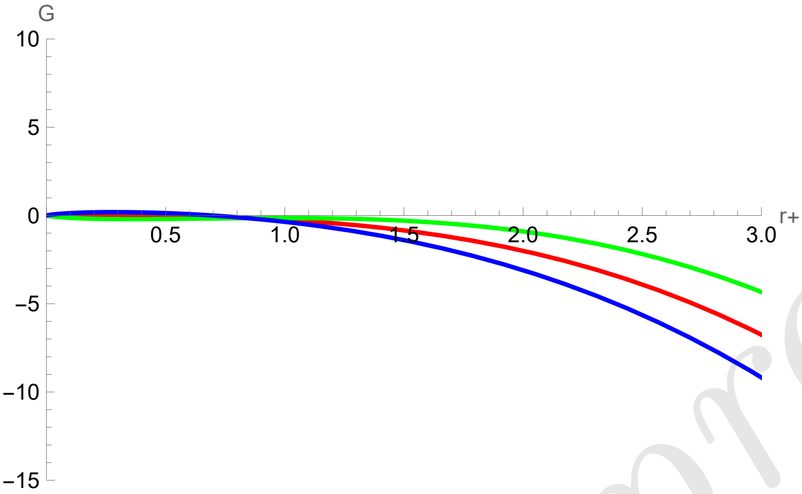

Figure 5: Gibbs free energy G versus black-string horizon radius r_+. Curves correspond to various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

6 Stability Criteria and Compressibility of Black Strings

In order to investigate phase transitions and thermodynamic stability during Hawking evaporation, we consider the specific heat capacity, which is defined by:

$$C = \left(\frac{\partial E}{\partial T_H}\right).$$

We obtain the following using Eqs. (2.5) and (5.2):

$$C = 2\beta + \alpha \pi r_+^2. \tag{6.1}$$

For $\beta = 0$, the heat capacity $C = \alpha \pi r_+^2$ is positive, signalling thermal stability. However, the heat capacity turns negative for $r_+^2 < -\frac{2\beta}{\pi\alpha}$ (i.e. $r_+^2 < \frac{2|\beta|}{\pi\alpha}$) when $\beta < 0$, indicating a thermodynamic instability. The mapping of stable and unstable regimes in Fig. 6 makes this transition very evident.

7 Final Remarks

The study of black strings within the framework of black hole thermodynamics reveals insight into the quantum structure of gravity. By extending classical black hole configurations to cylindrical geometries and embedding them in an Anti-de Sitter (AdS) spacetime, one obtains a tractable framework to probe the thermodynamic response under quantum corrections. In this work, we have examined such a black string system by incorporating modifications arising from the Generalized Uncertainty Principle (GUP), which encodes a minimal length scale motivated by various approaches to quantum gravity. The GUP framework leads to a deformation of the standard Heisenberg uncertainty relation, which in turn induces perturbative corrections to the Bekenstein–Hawking entropy. At leading order, these manifest as logarithmic corrections to the entropy–area relation. Interestingly,

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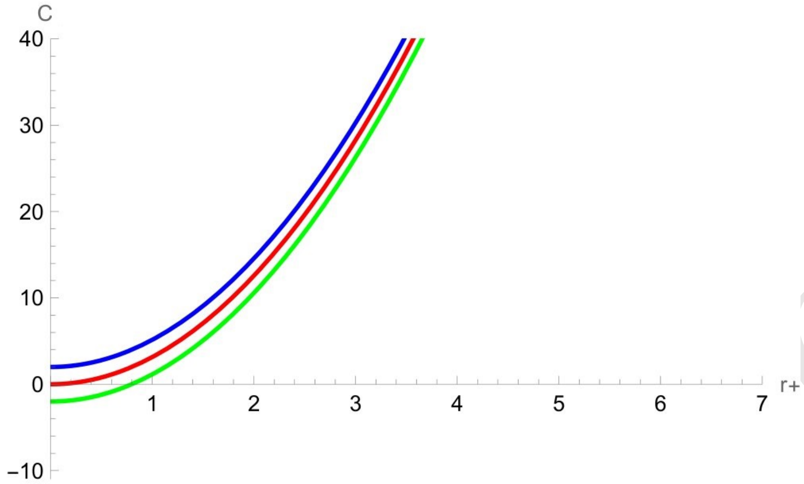

Figure 6: Heat capacity C versus black-string horizon radius r_+. Curves correspond to various values of the fluctuation parameter β: red for $\beta = 0$, blue for $\beta = 0.5$, and green for $\beta = -0.5$.

the corrected entropy may become negative in certain regimes, suggesting a possible repulsive gravitational effect at small scales—a phenomenon consistent with expected quantum gravitational behavior in the deep-quantum (sub-Planckian) regime.

Our analysis of thermodynamic potentials demonstrates that the corrected Helmholtz free energy decreases monotonically with increasing horizon radius, recovering classical behaviour in the absence of GUP effects. Positive correction parameters enhance this decrease, indicating a tendency toward faster stabilisation in the small-radius regime. Conversely, negative correction parameters may increase the free energy and delay thermal equilibrium, indicating potential instability. The pressure, derived from the Helmholtz potential, remains nearly constant at large horizon radii but exhibits sharp deviations as the radius approaches zero. In particular, the pressure diverges for small r_+ under negative corrections, suggesting enhanced spacetime curvature and strong tidal effects in the final stages of evaporation. Gibbs free energy analysis supports these conclusions, showing that GUP corrections generally lower the thermodynamic potential and can drive the system toward instability at small scales. Positive corrections enhance thermal stability, while negative corrections exacerbate instability, especially in the deep quantum regime. These findings highlight the role impact of minimal length effects on black string stability and energy dynamics. Thermal stability is further investigated through the specific heat, whose sign change under GUP corrections signals the onset of instability in small black strings.

In conclusion, the cylindrical AdS black string emerges as a useful theoretical setting for exploring the intersection of gravity and quantum mechanics. The incorporation of GUP corrections reveals a range of novel thermodynamic behaviours—especially in the small-radius, non-equilibrium domain—motivating further theoretical study. Future studies could build on this framework to examine critical phenomena, dynamical stability under perturbations, and the influence of quantum corrections on quasinormal modes and phase transitions in extended black string systems.

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Authors' Contributions

All authors contributed equally to this work.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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