



Review article

Particle Physics: A Crash Course for Mathematicians

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Abstract. This introductory work combines bottom-up and top-down approaches towards understanding the underlying categorical structure of possible unifying theories descending from string theory. Guided by well-established developments in the realm of categorical algebraic geometry and topological holography, we explain why abelianisation could potentially lead to furthering the understanding of how to embed Beyond the Standard Model scenarios in supersymmetric setups.

Keywords: Quantum Field Theory; Topological Holography; Category Theory; Quiver Varieties; Abelianisation.

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1 Introduction

At first sight, the title chosen for this article might appear provocative, and the length of the paper is certainly not fair towards the rich and deep advancements made by the phenomenological community so far¹. However, we stress that our aim is that of introducing important open questions in the realm of Particle Physics in a mathematical language approachable by categorical algebraic geometers.

Actually, this work is really meant to be the first of a series of papers by the same author continuing from past developments, bridging the gap between Pure Mathematics and Theoretical Particle Physics achievements. The author's engagement with experts from both lines of research is what motivated this project in the first instance. In upcoming work, we will be developing this description in terms of algebraic and Hilbert series calculation in greater detail. For the moment, we anticipate that the purpose of this introductory article is mainly fourfold:

- To illustrate (some) key fundamental problems in Theoretical Particle Physics necessitating further mathematical understanding.
- Opening the scene to well-established techniques in representation theory and categorical algebraic geometry to help in this regard.
- For those who are not experts in the field, serves as an oversimplified explanation of the powerful formalism put forward by Braverman, Finkelberg, and Nakajima (BFN), [28,29]. In doing so, we will explicitly show the relation between their work, that of Freed, Moore, and Teleman (FMT), [30], and Teleman's, [31], which, to the best of our knowledge, has only previously been referred to in [32,33].
- Motivating Phenomenologists to engage more with Pure Mathematicians, fostering mutual inspiration for common aims.

The main objective of this paper is that of explaining the higher categorical structures that are needed for describing the invariants associated to specific supersymmetric quiver gauge theories, with a particular focus on dualities and their mutual relations in terms of higher-categories. We believe that deepening its understanding and generalisations thereof, could lead to uncharted corners of mathematics that could be used for embedding of the Standard Model (SM) of Particle Physics, [40] in possible supersymmetric unifying theories, [41].

The present work is structured as follows:

1. Section 2 explains the motivations and outline of the present and forthcoming papers. In doing so, we briefly introduce scattering amplitudes as crucial tools within the context of particle physics, explaining their connection with Hilbert series. We then briefly explain some of the key open questions in Theoretical Particle Physics necessitating further mathematical understanding, specifically the hierarchy problem, and the lightness of the Higgs mass (and its relation to Supersymmetry (SUSY), [41]).
2. Section 3 is mostly a revision of [30,42], and we use it at this stage for multiple reasons. First of all because it allows to introduce gauge theories and Lagrangians in a mathematical language, which in turn is suitable to make connection with the higher-categorical Symmetry Topological Field Theory (SymTFT) prescription put forward

¹For a detailed overview of the state of the art and major developments in the field, we refer the interested reader to the rich literature at our disposal, including [1–11], and [12–18] for slightly more theoretically-oriented perspectives in this regard, especially in relation with string theory, [19–27].

by Freed, Moore, and Teleman, (FMT), [30], in the context of relative quantum field theories (QFTs)².

3. The natural way of describing gauge theories is by means of quivers. The Standard Model itself admits such a representation. It is also the case, though, that quivers are crucial for describing certain algebraic varieties associated with highly supersymmetric setups. Section 5 is therefore devoted to explaining in a simplified way the well-established BFN construction leading to ring homologies. The theories in question are 3D $\mathcal{N} = 4$ supersymmetric quiver gauge theories. This section highlights the importance that categorical algebraic geometry plays in identifying ring homologies accounting for the complete spectrum of a given theory, therefore giving a complementary insight with respect to the Hilbert series alone. In so doing, we will highlight the importance of *abelianisation*, [43,44]. In doing so we will explicitly show the relation between their work, that of Freed, Moore, and Teleman (FMT), [30], and Teleman's, [31], which, to the best of our knowledge, has only previously been mentioned in [32,33].
4. From the discussion of the previous section, the concluding section, 6, paves the way to forthcoming work, where a more detailed analysis with regard to the algebraic structure and Hilbert series of (Beyond) the Standard Model scenarios will be carried out. Additional complementary directions addressed by the same author will be developing ring homologies for Moore-Tachikawa varieties, [34], beyond categorical duality³, and understanding their implications for Koszul duality, [35]. We plan to report on any advancements in this regard in the near future.

2 Roadmap

This first section is devoted to outlining the path we will be adopting for addressing the issues mentioned in the Introduction. The diagram of figure 2 exemplifies this, and we will start addressing the topics outlined there starting from section 5, continuing in an upcoming work. For those who are already familiar with the terminology adopted in figure 2, the first three sections of this paper are purely revision. However, they might still find some interesting connections, which, to the best of our knowledge, haven't been spelled out in the literature with the same emphasis and perspective.

Essentially, sections 2, 3, and 5 can be thought of as an extremely concise summary of the basic knowledge gathered by a mathematical physicist in the attempt to explain a possible roadmap towards deepening the understanding of some major open questions in Theoretical Particle Physics from a Mathematician's point of view.

As an advanced notice, we advise the reader not to be misguided by the oversimplified explanation of the first few pages of this work: technicalities will soon arise.

In this preliminary section, we introduce the unfamiliar reader to:

1. The basic building blocks of QFT⁴.
2. Some key theoretical tools adopted by Phenomenologists for analysing them.

²As explained in Section 3, SymTFTs can essentially be thought of as realisations of topological holography.

³Thereby following up from the analysis carried out in [36].

⁴We strongly suggest the interested reader to refer to the work of pioneers in the field for a complete treatment of QFT, specifically [37–39].

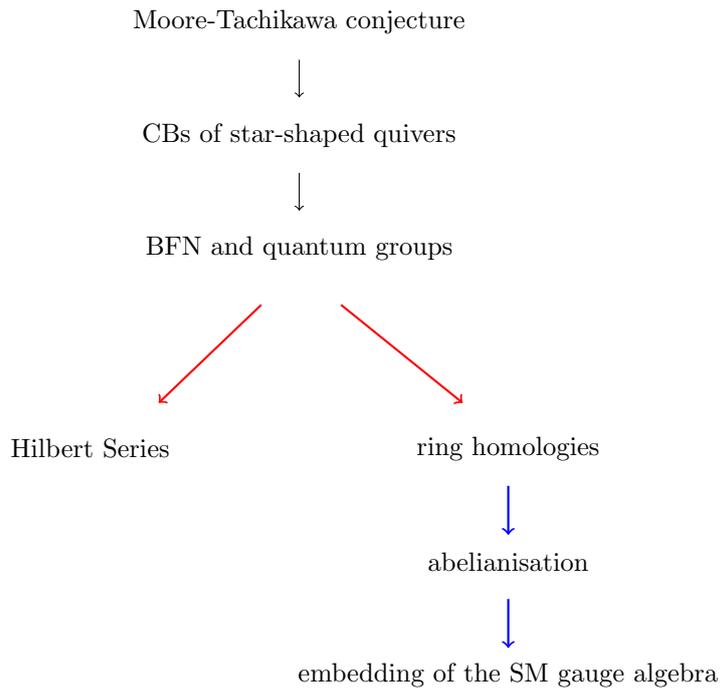


Figure 1: Roadmap for the present and upcoming work.

2.1 Fields and Interactions

Quantum fields are mutually-interacting entities exhibit wave-particle duality. QFT is arguably the most successful scientific theory so far, with the Standard Model (SM) of Particle Physics being a prototypical example. In some cases, QFT predicts to great accuracy observational outcomes, and has also been used in the context of pure mathematics.

With the advent of the quantum revolution, fields were promoted to being fundamental constituents, whereas all fundamental particles were understood to arise as a quantum mechanical (QM) effect. This extends to all particles known so far, namely quarks, leptons, gluons, W- and Z-bosons, and the Higgs boson.

The SM is an example of a QFT containing all the particles known up to now, with their associated three force fields⁵ Extensions of the SM, referred to as Beyond the Standard Model (BSM) setups, are currently an active area of research, providing extensions of the SM by adding interactions and/or particles.

A given QFT is defined in terms of a Lagrangian density, which, upon integration over spacetime, defines the action, S , which is, essentially, a functional of the fields. The Variational Principle constraints the action in such a way that its extremisation leads to the equations of motion of the respective fields. Importantly, the action is needed for defining the partition function, i.e., the expectation value of the identity, as well as any other expectation value of operators⁶ accounted for by a given effective field theory (EFT).

⁵The matter fields comprised in it come in 3 families, plus the Higgs boson.

⁶The latter are also known as correlation functions.

2.2 Amplitudes and Hilbert series

2.2.1 Amplitudes

Scattering amplitudes are among the most important tools in Particle Physics. They are the building blocks for defining transition amplitudes of a certain set of states to another. In the context of collider physics, they are an essential theoretical background tool for predicting and analysing particle collisions and decays. Importantly, if new physics were to be discovered, it must be signalled by a mismatch between experimental outcomes and scattering calculations, that could either feature as the detection of new particles, and/or the identification of a new interaction (and therefore a new fundamental force).

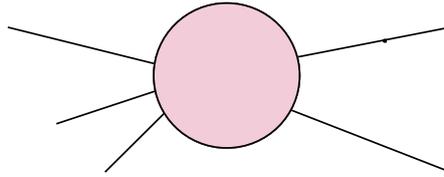


Figure 2: A simplified depiction of a scattering process. The S -matrix defines the amplitude from the incoming states (black lines on the left) to the outgoing states (black lines on the right). The shaded circle in the middle encodes the information regarding the interactions and operator content characterising a certain EFT.

In a nutshell, if our theoretical knowledge of particle physics were complete, we should be able to explain all experimental outcomes, recasting them in the Lagrangian theories at hand. However, it so happens that there are some crucial unanswered questions, that clearly signal the need for a deeper mathematical understanding of the models developed so far. Those that motivated our work in the first place are the following:

1. The mass hierarchy problem.
2. The lightness of the Higgs mass.

To help the unfamiliar reader, we will come back to explaining these issues in section 2.3.

2.2.2 Hilbert series and effective operators

$$\text{HS}[\phi] = \prod_{i=1}^n \int_{\mathcal{G}_i} d\mu_i \text{PE}[\varphi, R], \quad (2.1)$$

where \mathcal{G}_i denotes a given gauge group, φ is a *spurion* variable⁷, $d\mu_i$ denotes the Haar measure, and $\text{PE}[\varphi, R]$ the *plethystic exponential*, which is defined as follows

$$\text{PE}[\phi, R] = \exp \left[\sum_{r=1}^{\infty} \frac{\phi^r \chi_R(z_j^r)}{r} \right], \quad \text{PE}[\psi, R] = \exp \left[\sum_{r=1}^{\infty} (-1)^{r+1} \frac{\psi^r \chi_R(z_j^r)}{r} \right], \quad (2.2)$$

for bosons and fermions, respectively. We will be entering the details of the calculation of (2.1) in an upcoming work. For the purpose of this introductory work, it is enough to know that HS is constructed in terms of the degrees of freedom (namely the particle content) and

⁷A spurion could either be a scalar, a fermion, or a gauge field.

their transformation properties under the symmetries of a given theory. This is enough to say that, in order to build the Hilbert series, we need to know what the complete set of operators of a given theory is. Completeness of the spectrum of operators is a major sought-after objective in particle physics, since it paves the way towards identifying possible extensions of the fundamental laws of nature. Given the relation between scattering amplitudes and HS, a mismatch with predictions for a given theory signal the need to extend the effective field theory of reference. If the theory in question is taken to be the Standard Model (SM) of Particle Physics, its resulting extensions are referred to as Beyond the Standard Model (BSM) effective field theories, [46–48].

2.3 (Some) Open Questions in Quantum Field Theory

Albeit having been successfully verified experimentally, the SM of Particle Physics is clearly incomplete. The Higgs boson, [50,51], responsible for the mass of fundamental particles, was the last piece of a jigsaw to be found, yet the first of a more complicated one. The Higgs boson is a scalar responsible for the mass gain of fundamental particles, via the so-called Higgs mechanism⁸. Undoubtedly one of the major theoretical physics achievements of the past century, the model is clearly incomplete. The two major issues we indicate are the mass hierarchy problem and the lightness of the Higgs boson mass. Each one of them is the object of active investigation by world-leading experts in the field, and by no means we are attempting to provide a full explanation of either of them in our treatment. Nevertheless, we consider them as part of the motivations leading to the need for furthering the understanding of the underlying mathematical structure of QFT.

2.3.1 The Mass Hierarchy Problem

Essentially, this is a statement regarding the unexplained discrepancy between the energy scales at which the fundamental interactions decouple. A successful unifying theory should be able to explain why this symmetry breaking pattern.

2.3.2 Lightness of the Higgs Boson and Supersymmetry

Another open question highlighting the fact that the SM is still incomplete is that the Higgs mass is too light. Given that it is the only spinless particle in the SM, theorems imply it should be extremely massive. One of the possible reasons explaining the very light mass of the Higgs was thought to be Supersymmetry (SUSY), proposing that, in order to compensate the very light mass observed, new particles (referred to as superpartners) should be detected at the Large Hadron Collider (LHC). Up to now, though, these superpartners haven't been observed by collider experiments at the energies predicted by the theory. This inevitably leads to the question, to what extent is SUSY able to provide a consistent theory of everything, given that String Theory so strongly relies upon it?

We believe that SUSY and String Theory are very promising candidates for a unifying theory of everything, but, in order to overcome the apparent shortcomings listed above, we need to address the deep question highlighted in the introduction, namely: "What is (the mathematical structure of) a QFT?". The present work is meant to provide further insights in addressing this question, by making use of mathematical advancements in the field of categorical algebraic geometry. Our findings shed light on a new perspective that is proposed to be a promising alternative top-down construction of QFTs. The ultimate aim

⁸We refer the interested reader to the original work by Professor Peter Higgs for a detailed explanation of the process, [50,51].

would be that of identifying the mathematical structure underlying the SM, and, with it, its embedding in the most suitable string theory vacuum possible, to which we plan to report in upcoming work.

3 Quantum Field Theory from Category Theory

We now turn to explaining how QFT can be mathematically described in terms of Category Theory. In doing so, we will also be introducing the notion of moduli space, as denoting the minima of the theory in question, namely the least energy field configurations. Albeit the sample theories presented in this section are quite trivial, they are sufficient for building intuition for what lies ahead in our treatment.

To begin with, we start off by listing some of the ingredients that are most relevant to our analysis⁹, briefly outlined in the table below.

<u>Particle Physics</u>	<u>Category Theory</u>
Gauge Theory	Algebras
Operators	Homologies
Quivers	Quivers/Algebraic varieties
Hilbert Series	Hilbert Series
SUSY	SUSY

This shows some quantities/entities encountered in the realm of Particle Physics and their Category Theory counterpart. We have highlighted two of them in red. The reason for this will be more clear in section 5, but the familiar reader will certainly notice that some motivation was already hinted at in section 2. For the moment, we simply recall what was mentioned in the Introduction, namely that the Hilbert series and ring homologies are key complementary ingredients towards understanding a given QFT. Among the two, though, the latter is the one that is more directly related to understanding the underlying categorical structure of the theory in question, and therefore constitutes the ultimate objective one wishes to obtain. On the other hand, the Hilbert series, is obtained from ring homologies, but hides some key categorical features that are only accessible via the homological perspective.¹⁰

This intermediate section is mostly an overview of [30,30,42], and is needed in order to understand the analysis carried out in the concluding part of our work, 5. This section is structured as follows:

1. At first, we review mathematical gauge theory and the definition of moduli spaces therein, [30].
2. Then, we recall the notion of relative field theories, [42], which can also be adapted to gauge theories.
3. In conclusion, we explain how this fits with the higher-categorical prescription of Freed, Moore and Teleman, [30], and how gauging can be implemented in terms of condensable

⁹As we shall see, more richness will be added in due course in our treatment. The aim of the table is simply that of helping the reader in gaining intuition for the setup and the motivations behind the procedure adopted.

¹⁰Indeed, it is known that different ring homologies could lead to the same Hilbert series, [103].

homomorphisms in the Symmetry Topological Field Theory categorising the symmetries of a given QFT, [53], thereby providing an example of topological holography, whose boundary is a codimension-1 relative QFT, and the bulk is a Chern-Simons theory.

For reasons that will become clear by the end of this section, the crucial part of the higher-categorical perspective can be summarised as follows:

- Keeping track of the identity.
- Embedding the gauged algebra.

Most importantly, they both lie within the notion of *abelianisation*. In section 5 we will see explicitly how they arise within the context of supersymmetric quiver varieties, [43,44].

3.1 Principal bundles and connections

Formulating gauge theories in Lagrangian formulation that is suitable for a QFT requires some preliminary differential geometry notation, which we will now overview.

The initial ingredients we start from are a manifold, M , and a Lie group, G . Then, a *principal G -bundle*, $P \rightarrow M$ is defined as a manifold P on which G acts freely on the right with quotient

$$P/G \simeq M, \quad (3.1)$$

such that there exist local sections. If P', P are principal G -bundles over M , then an isomorphism

$$\varphi' : P' \longrightarrow P, \quad (3.2)$$

is a smooth diffeomorphism which commutes with G , and induces the identity map on M . If $P \equiv P'$, such an automorphism defines a gauge transformation of P .

For every M , there is a category of principal G -bundles and isomorphisms.

A *connection* on a principal G -bundle $\pi : P \rightarrow M$ is a G -invariant distribution in TP which is transverse to the vertical distribution $\ker \pi$. A connection can therefore be expressed as a 1-form $A \in \Omega^1(P; \mathfrak{g})$ whose value at p is the projection

$$T_p P \longrightarrow V_p \simeq \mathfrak{g}, \quad (3.3)$$

with kernel H_p .

Connections form a category, arranging in a set of equivalence classes under isomorphisms.

3.2 Mathematical Gauge theory

The prototypical example of a gauge theory is Maxwell's theory. In its quantised formulation, one needs to introduce Dirac's charge quantisation, replacing the group \mathbb{R} by the compact group $\mathbb{Z}/2\mathbb{Z}$. This statement can be generalised by replacing $\mathbb{Z}/2\mathbb{Z}$ with any compact Lie group G .

Hence, we are drawn to the conclusion that the Lagrangian formulation of the field theory in question is simply based on the following data:

1. A compact Lie group with Lie algebra, \mathfrak{g} .
2. A bi-invariant inner product on \mathfrak{g} , $\langle \cdot, \cdot \rangle$.

From this, the total Lagrangian for the gauge field reads

$$L = -\frac{1}{2} \langle F_A \wedge \star F_A \rangle \quad (3.4)$$

from which the energy density reads

$$\mathcal{E}(A) = \left[\sum_{\mu < \nu} \frac{1}{2} |F_{\mu\nu}|^2 \right] |d^{n-1}x|, \quad (3.5)$$

whose moduli space is simply

$$\mathcal{M}_{vac} = \text{pt}, \quad (3.6)$$

where the calculation is understood to be performed on the space of equivalence classes of connections.

3.2.1 The gauged sigma-model

This is a very general bosonic field theory, combining pure gauge theory with the σ -model, both of which are greatly used in QFT. The data specifying the model in question are:

1. A Lie group, G , with Lie algebra, \mathfrak{g} .
2. A bi-invariant scalar product, \langle, \rangle , on \mathfrak{g} .
3. A Riemannian manifold, X , on which G acts by isometries.
4. A potential function

$$V : X \longrightarrow \mathbb{R}, \quad (3.7)$$

invariant under G .

The space of fields, \mathcal{F} , consists of pairs

$$\mathcal{F} \stackrel{def.}{=} \{(A, \phi)\}, \quad (3.8)$$

with A denoting a collection of principal G -bundles $P \rightarrow M$, and ϕ a section of the associated bundle over $P \times_G X \rightarrow M$. In most cases, it is convenient to view ϕ as an equivariant map

$$\phi : P \longrightarrow X. \quad (3.9)$$

The space of fields can be thought of as a category, with an isomorphism

$$\varphi : P' \longrightarrow P, \quad (3.10)$$

of principal bundles, inducing an isomorphism of fields

$$(A', \phi') \longrightarrow (A, \phi). \quad (3.11)$$

The global symmetry group of the theory is the subgroup of isometries of X commuting with the G -action, and preserving the potential, V .

The Lagrangian density, therefore reads

$$\mathcal{L} = -\frac{1}{2} |F_A|^2 + \frac{1}{2} |\mathcal{D}_A \phi|^2 - \phi^* V, \quad (3.12)$$

where \mathcal{D}_A denotes the covariant derivative, coupling ϕ with A . The energy density of the pair (A, ϕ) , can readily be extracted from (3.12), and reads

$$\mathcal{E}(A, \phi) = \left[\sum_{\mu < \nu} \frac{1}{2} |F_{\mu\nu}|^2 + \sum_{\mu} \frac{1}{2} |(\partial_A)_\mu \phi|^2 + \phi^* V \right] |d^{D-1}x|, \quad (3.13)$$

where D denotes the total number of spacetime dimensions.

Assuming V has a minimum at 0, one can look for solitons with 0-energy. By direct inspection of (3.13), it is possible to deduce that:

1. The first term implies that A is flat, hence, up to equivalence, it is the trivial connection with zero-curvature.
2. The second term tells us that ϕ has to be a constant for it to be a zero-energy solution.
3. The third term constraints the constant value of ϕ to lie in the $V^{-1}(0)$ set.

A trivial connection A has a group of automorphisms that are isomorphic to G , i.e., the group of global gauge transformations. For a vacuum solution, ϕ must be a constant function with values in $V^{-1}(0)$, and, therefore, $V^{-1}(0)$ inherits the same G -action from ϕ . Hence, the moduli space of vacua is a subquotient space of X

$$\mathcal{M}_{vac} \stackrel{def.}{=} V^{-1}(0) / G. \quad (3.14)$$

For the case of certain supersymmetric field theories, X is Kähler or hyperkähler, and $V^{-1}(0)$ is the norm square of an appropriate moment map, in which case \mathcal{M}_{vac} reduces to a Kähler or hyperkähler quotient (with the latter actually being the setting where this had first been studied).

3.3 Relative Field Theories

Symmetries in field theory are expressed in terms of topological defects acting as operators on state spaces of the theory in question. The idea at the heart of this is the notion of abstract groups and abstract algebras. In their work, [30], Freed, Moore and Teleman propose an abstract symmetry structure in the context of field theory and its concrete realisation.

A field theory is analogous to a linear representation of a Lie group or a module over an algebra. Hence their definition of the boundary theory as a module. The essential content of their definition is that of expressing a QFT, F , as a sandwich $\rho \otimes_\sigma \tilde{F}$, with ρ a regular Dirichlet BC (right boundary theory for σ) and \tilde{F} the left boundary theory for σ (typically not topological).

Altogether, (σ, ρ) constitutes the n -dimensional *quiche*, defining the abstract symmetry structure of a given QFT. Defects away from \tilde{F} are topological and belong to (σ, ρ) . (σ, ρ) -defects are analogous to elements of an abstract algebra (connection with GKS). The novelty of the treatment by Freed, Moore and Teleman is the fact that the sandwich representation develops a calculus of topological defects acting on a QFT (based on fully local TFT). The TFT imposes strong finiteness constraints on the QFT.

Fixing $N \in \mathbb{Z}^{\geq 0}$, then a quiche is a pair (σ, ρ) in which

$$\sigma : \text{Bord}_{N+1}(F) \rightarrow \mathcal{C}, \quad (3.15)$$

is an $N + 1$ -dimensional TFT and ρ is a right topological σ -module. The quiche is N -dimensional, hence it shares the same dimensionality as the theory on which it acts. Let F be an N -dimensional field theory. A (σ, ρ) -module structure on F is a pair (\tilde{F}, θ) , in which \tilde{F} is a left σ -module and θ is an isomorphism

$$\theta : \rho \otimes_{\sigma} \tilde{F} \xrightarrow{\cong} F, \tag{3.16}$$

of absolute N -dimensional theories. The LHS of (3.35) defines the dimensional reduction, i.e., the sandwich we were referring to earlier. σ needs only be a *once-categorified* N -dimensional theory, whereas ρ and \tilde{F} are relative theories.

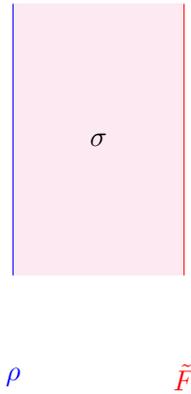


Figure 3: Specifying the topological boundary conditions turns a relative QFT into an absolute one.

Gauge theories are examples of relative theories, [42].

More generally, *relative theories* are models whose fields form a fibration

$$\mathcal{F} \longrightarrow \mathcal{M} \xrightarrow{\pi} \mathcal{B}, \tag{3.17}$$

where \mathcal{B} and \mathcal{F} denote the base and the fiber, respectively. Relative fields are the fibers of π . The path integral over the base reduces a finite sum. For the theory to be absolute, the same should hold for \mathcal{F} as well.

As originally outlined in [42], QFT F relative to α is a homomorphism

$$\tilde{F} : \tau_{\leq n} \alpha \rightarrow \mathbf{1}, \tag{3.18}$$

where α is the bulk theory w.r.t. F .

Let G be a finite group. A classifying space BG is derived from a contractible topological space EG equipped with a free G -action by taking the quotient. The homotopy type of BG is independent of choices. If X is a topological space equipped with a G -action, then the Borel construction is the total space of the fiber bundle.

$$\begin{aligned} X_G &\equiv EG \times_G X \\ \downarrow \pi & \\ BG & \end{aligned} \tag{3.19}$$

with fiber X . If $* \in BG$ is a point, and we choose a basepoint in the G -orbit in EG labelled by $*$, then the fiber $\pi^{-1}(*)$ is canonically identified with X . The abstract group symmetry data is the pair $(BG, *)$, and a realisation of the symmetry $(BG, *)$ on X is a fiber bundle over BG together with an identification of the fiber over $* \in BG$ with X .

Following [42], we outline two preliminary examples of relative field theories:

1. Sigma models.
2. Gauge theories.

3.4 Sigma models

Data: G a finite group and M a smooth manifold with a free left-acting G and quotient \bar{M} (equivalently the gauged σ model on M).

Denoting with $BG(X)$ the collection of principal G -bundles over X ,

A principal G -bundle is a covering space $P \rightarrow X$ and a free- G action on P such that $P \rightarrow X$ is a quotient map for the G -action. We denote the collection of principal G -bundles over X with $BG(X)$, with the latter being a groupoid, and, as such, is a category in which every morphism is invertible.

A *symmetry*

$$\varphi : (P \rightarrow X) \rightarrow (P' \rightarrow X), \quad (3.20)$$

is a diffeomorphism

$$\varphi : P \rightarrow P', \quad (3.21)$$

commuting with G and covering id_X . The automorphism group of $P \rightarrow X$ is the group of gauge transformations.

The path-integral over $BG(Y)$ is an integral over the equivalence classes of G -bundles.

Fixing $(P \rightarrow X) \in BG(X)$, a relative field over $(P \rightarrow X)$ is a pair (f, θ) consisting of a smooth map $f : X \rightarrow \bar{M}$ and an isomorphism $\theta : (P \rightarrow X) \rightarrow f^*(M \rightarrow \bar{M})$ of G -bundles over X . Given that relative fields feature no automorphisms, they are *rigid*, and therefore constitute a space rather than a category.

Given a pair of theories, (α, F) , with α topological and defined on all manifolds, and F a Riemannian theory, using the knowledge of the σ -model F to predict the structure of α .

The relative path-integral $F(X)$ on a closed D -dimensional manifold is an integral over relative fields, hence a function

$$F(X) : BG(X) \rightarrow \mathbb{C}, \quad (3.22)$$

As such, $F(X)$ is invariant under symmetries of $BG(X)$, and is therefore expressed as a function on equivalence classes, $H^1(X; G)$, with the latter denoting the isomorphic classes of principal G -bundles over X . From this, (3.22) can be re-expressed as

$$F(X) : H^1(X; G) \rightarrow \mathbb{C}, \quad (3.23)$$

If F is to be a QFT relative to a $D + 1$ -dimensional bulk α , in the sense that

$$F(X) : \mathbb{C} \rightarrow \alpha(X), \quad (3.24)$$

then, $F(X)$ can be identified as an element of the vector space $\alpha(X)$, leading to the conclusion that $\alpha(X)$ is the free vector space on the finite set $H^1(X; G)$.

Next, considering a closed $D - 1$ -dimensional manifold, Y , the relative canonical quantisation of $F(Y)$ is obtained by performing the quantisation on the fibers of the map

$$\text{Map}(Y, \bar{M}) \rightarrow BG(Y), \tag{3.25}$$

such that the vector bundle is

$$F(Y) \rightarrow BG(Y). \tag{3.26}$$

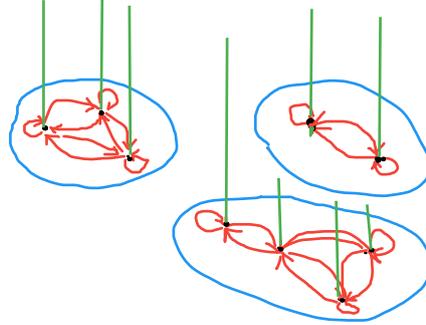


Figure 4: A depiction of the vector bundle.

The bundles are grouped into isomorphism classes. Points featuring in each isomorphism class denote the G -bundles $Q \rightarrow Y$, and the arrows the isomorphisms between the G - bundles. Selecting a representative bundle, $Q \rightarrow Y$, in each isomorphism class, the grupoid $BG(Y)$ corresponds to a simpler grupoid with a finite set of $H^1(Y; G)$ and no morphisms between them.

If $m \in H^1(Y; G)$ is the class of a G -bundle $Q \rightarrow Y$, then the automorphism group of m is the group of gauge transformations of $Q \rightarrow Y$, and the equivariant bundle, (3.26). decomposes into topological vector spaces indexed by pairs (m, e) in which $m \in H^1(Y; G)$ and e is a complex irreducible representation of the automorphism group of m . The partition function on Y

$$F(Y) : \mathbf{1}(Y) \rightarrow \alpha(Y), \tag{3.27}$$

may be identified with an object belonging to the linear category $\alpha(Y)$, leading to the prediction that $\alpha(Y)$ is the free Vect_{top} -module with basis pairs (m, e) .

For the case in which X is a 1D theory, namely $X \equiv S^1$, $H^1(S^1; G)$ is the set of *conjugacy classes* in G . If $Y \equiv \text{pt}$, then $H^1(\text{pt}; G)$ has a single element, representing the trivial bundle with automorphism group G . Consequently, $\alpha(\text{pt})$ defines the category of representations of G .

In this case, α is the finite 2D gauge theory with gauge group G , and can be defined on any manifold of $D \leq 2$ by a finite path-integral. Given that the theory has a vanishing Lagrangian, the partition function on a closed 2D manifold, M_2 , reduces to the weighted counting of G -bundles

$$\alpha(M_2) = \sum_{[R \rightarrow M_2]} \frac{1}{|\text{Aut}(R \rightarrow M_2)|}. \tag{3.28}$$

The finite path integral over $X \equiv S^1$ is the space of invariant sections of the trivial equivariant line bundle $BG(S^1)$. On $Y \equiv \text{pt}$, instead, the exponential of the action has constant

value in the linear category Vect_{top} , hence the path integral defines the subcategory of invariants under G , namely the category of representations of G .

3.4.1 From relative to absolute

An *absolute* theory can be recovered from a relative one under gauging by a finite subgroup $G' \subset G$. In such case, the σ -model on a closed D -dimensional manifold can be expressed in terms of the relative theory as follows

$$f_{G'}(X) = \sum_{m' \in H^1(X; G')} \frac{1}{|Z_{G'}(m')|} F(X; m'), \quad (3.29)$$

where the sum runs over the equivalence classes of G' -bundles $P \rightarrow X$, and $Z_{G'}(m')$ denotes the automorphism group of a representative of the equivalence class m' .

Following similar arguments, the quantum topological vector space on a closed $D - 1$ -dimensional manifold Y can be expressed as

$$f_{G'}(Y) = \bigoplus_{m' \equiv [Q' \rightarrow Y] \in H^1(X; G')} F(Y; Q \rightarrow Y)^{\text{Aut}(Q' \rightarrow Y)}. \quad (3.30)$$

3.5 Gauge theories

A very similar analysis holds for the case of gauge theories, with all the fields' categorical levels enhanced by 1 unit, due to the presence of an additional symmetry.

In this case, the theory is specified by the following data: a Lie algebra, \mathfrak{g} , a covering homomorphism $G \rightarrow \tilde{G}$ of compact connected Lie groups with kernel G_o a finite central abelian subgroup of G . For example, if $G_o \subset G$ and $\tilde{G} \equiv G \setminus G_o$ is the adjoint group, this gives a canonical choice of data associated to a compact semisimple Lie algebra.

Assuming $\tilde{P} \rightarrow X$ denotes a principal \tilde{G} bundle, the obstruction to lifting to a principal G -bundle is measured by a G_o -gerbe, $\mathcal{G}(\tilde{P}) \rightarrow X$, and with $B^2 G_o(X)$ the collection of G_o -gerbes over X . $B^2 G_o(X)$ is a 1-category. Its equivalence classes constitute the cohomology group $H^2(X; G_o)$. The group of equivalence classes of automorphisms between objects is $H^1(X; G_o)$, and the group of automorphisms of automorphisms is $H^0(X; G_o)$.

For gauge theories, a relative field can be defined as $(\mathcal{G} \rightarrow X) \in B^2 G_o(X)$. A relative field over $(\mathcal{G} \rightarrow X)$ is a pair $(\tilde{\Theta}, \theta)$ consisting of a \tilde{G} -connection $\tilde{\Theta}$ and an isomorphism $\theta : \mathcal{G} \rightarrow \mathcal{G}(\tilde{\Theta})$ of G_o -gerbes. $(\tilde{\Theta}, \theta)$ form an ordinary groupoid, with no automorphisms between automorphisms. The fields are \tilde{G} -connections with G -gauge transformations.

The quantum theories (α, F) that can be built from these fields are a $D + 1$ dimensional topological theory and a D -dimensional relative theory F . α is now defined as a path-integral over the G_o -gerbes, specifically

$$\alpha(M_2) \equiv \frac{|H^2(X; G_o)| |H^0(X; G_o)|}{|H^1(X; G_o)|}. \quad (3.31)$$

Consequently, $\alpha(X)$ is the vector space of complex-valued functions on $B^2(X)$. The relative theory F is an element of $\alpha(X)$ since

$$F(X) \equiv H^2(X; G_o) \rightarrow \mathbb{C}, \quad (3.32)$$

$\alpha(Y)$, instead, defines the linear category of vector bundles over $B^2 G_o(Y)$. The relative theory F determines a particular vector bundle $F(Y) \rightarrow B^2 G_o(Y)$.

For any finite abelian group A , its Pontryagin dual is defined as follows

$$A^\vee \stackrel{\text{def.}}{=} \text{Hom}(A, G_o). \tag{3.33}$$

Once having chosen the basepoints, $F(Y) \rightarrow B^2 G_o(Y)$ determines the topological vector spaces $F(Y; m, e)$ for $m \in H^2(Y; G_o), e \in H^1(Y; G_o)^\vee$, where (m, e) denote the classes of discrete magnetic and electric fluxes.

3.5.1 From relative to absolute

Once more, an absolute gauge theory can be obtained by gauging by a subgroup of G , leading to a relic gauge group $G \setminus G'_o, G'_o \subset G_o$, and its expression reads

$$f_{G'_o}(X) = \sum_{m' \in H^2(X; G'_o)} \frac{|H^0(X; G'_o)|}{|H^1(X; G'_o)|} F(X; m'). \tag{3.34}$$

In the formulation of [42], a *relative* field theory, \tilde{F} , requires additional topological data in order to be fully specified. Such data is encoded in a pair (σ, ρ) , referred to as *quiche*. σ is the *symmetry topological field theory* (SymTFT), whereas ρ geometrises the choice of boundary conditions for the fields defining the relative theory \tilde{F} . The overall system, depicted in figure 5, gives rise to an *absolute* QFT, F_ρ .

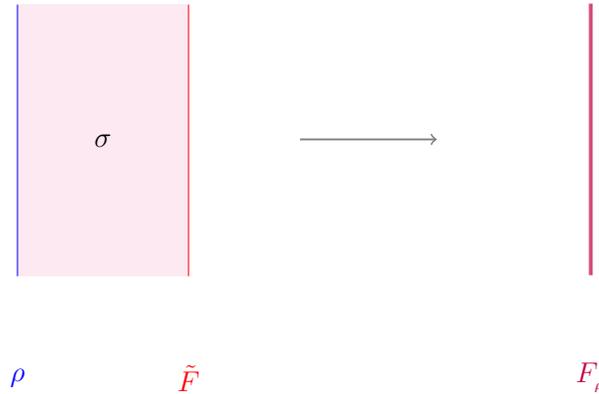


Figure 5: The Freed-Moore-Teleman setup, with \tilde{F} denoting a relative QFT. Specifying the topological data (σ, ρ) , the resulting theory, F_ρ is absolute.

In mathematical terms, the description outlined above can be formulated in terms of bordism in the following way. Fixing $N \in \mathbb{Z}^{\geq 0}$, then a quiche is a pair (σ, ρ) in which $\sigma : \text{Bord}_{N+1}(F) \rightarrow \mathcal{C}$ is an $N + 1$ -dimensional TFT and ρ is a right topological σ -module. The quiche is N -dimensional, hence it shares the same dimensionality as the theory on which it acts. Let F be an N -dimensional field theory. A (σ, ρ) -*module structure* on F is a pair (\tilde{F}, θ) , in which \tilde{F} is a left σ -module and θ is an isomorphism

$$\theta : \rho \otimes_\sigma \tilde{F} \xrightarrow{\cong} F_\rho, \tag{3.35}$$

of absolute N -dimensional theories, with (3.35) defining the dimensional reduction leading to the absolute theory. σ needs only be a *once-categorified* N -dimensional theory, whereas ρ and \tilde{F} are relative theories.

Having explained the notion of relative QFTs, we now provide some further explanation for what gauging a categorical structure actually means. In doing so, we refer to the work of many experts in the field, and, in particular [53]. As explained in such reference, for any fusion n -category \mathfrak{G} , any fiber functor

$$\mathcal{F} : \mathfrak{G} \rightarrow \mathbf{nVec}, \quad (3.36)$$

selects \mathbf{nVec} as the image of a condensation algebra living in \mathfrak{G} , corresponding to a projection on the identity. The gauging process, can therefore be defined as a map

$$\mu : \mathfrak{G} \rightarrow \mathcal{A}_C, \quad (3.37)$$

with \mathcal{A}_C the algebra of invertible topological operators in \mathfrak{G} . Given (3.37), the norm element

$$N \stackrel{def}{=} \bigoplus_{g \in \mathfrak{G}} \mu(g). \quad (3.38)$$

carries the structure of an n -categorical idempotent, also known as *condensation algebra*, depicted in black in figure 6. The requirement for (3.38) to be a higher-idempotent is needed to ensure the flooding doesn't depend on the specific features of the network being adopted to perform the gauging. The algebra of topological operators that are left are denoted by $\mathcal{A} //^\mu \mathfrak{G}$. The equivalence of the second and third picture from the left in figure 6 follows from N being idempotent. This pattern emerges, for example, when gauging the 5D SymTFT of class \mathcal{S} theories, with the objects of the 2-category in question being Wilson surfaces charged under the 1-form symmetry being gauged.

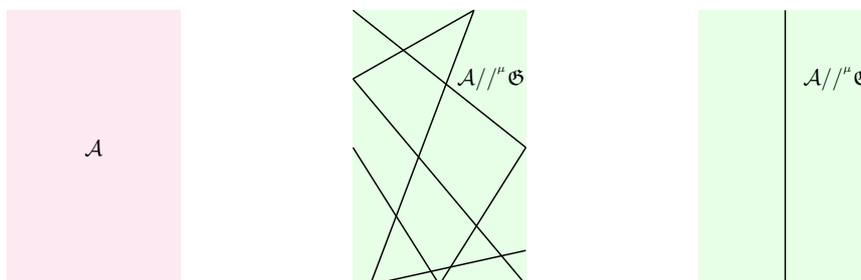


Figure 6: Gauging corresponds to condensing an algebra in a TFT. Idempotency ensures the resulting theory can be effectively thought of as featuring a unique defect, as shown on the RHS.

4 Key point

The aim of the analysis carried out in the present and following section is that of explaining how ring homologies are actually related to higher-categorical structures in the SymTFT construction of [30].

The key points to bare in mind for the following section can be summarised in terms of two mutually related points:

1. The need to keep track of the identity of the gaugeable algebra. Indeed, the moment map (3.37) is essentially providing a redefinition of the identity once the gauging has been performed¹¹.
2. The embedding of the gauged algebra.

By the end of section 3 it will become clear how both points are related to abelianisation, [43,44], and how this is crucial for accessing the mathematical structure of a given quiven QFT.

5 Quivers

Equipped with the tools revised in the previous sections, we are now ready to begin the analysis outlined in the roadmap illustrated in figure 2. The top entry of our diagram reads an example of a 2-category, namely that of Moore-Tachikawa varieties¹². The reason why we are referring to them as our main example is their relation to Coulomb branches of star-shaped quivers, [43], to which we can apply the analysis of section 3. The setups we are dealing with are highly supersymmetric, thereby describing theories that are very far from the SM of particle physics, even in its minimally supersymmetric realisation (MSSM). However, the purpose of this article is that of highlighting features in categorical structures of supersymmetric gauge theories that could come to aid in furthering the understanding of extensions of the SM to possible completions in unifying theories, in turn descending from a superymmetric parent setting, for the reasons outlined in section 2. Furthermore, it is important to remark that, the quiver description of the SM and its extensions is an essential step towards embedding them in any string theory setup, and we refer the reader to some key works on this regard, such as [13,54].

In section 3, we have gathered sufficient evidence supporting the utility of applying category theory for describing simple examples of QFT models. We now switch gears, and turn to a more algebro-geometric analysis of the setup outlined so far. In doing so, we will mostly be relying on [28,29,43,44], shedding new light and emphasising techniques that should be further developed and applied to Beyond the Standard Model setups¹³.

The present section is structured as follows:

1. At first, we overview some key features of equivariant cohomology and the convolution product, both within the context of quiver gauge theories, [28,29]
2. Then, we introduce Moore-Tachikawa varieties and their 2D TFT counterpart¹⁴, connecting categorical structures and quiver varieties, [34]. Making use of [43], we emphasise how and when 3D mirror symmetry manifests from a categorical perspective, specifically in terms of bordisms, making use of the Kostant-Whittaker symplectic reduction, [28].
3. Last but not least, we close the section highlighting the importance of abelianisation for embedding the ring homology of the theory in question, explaining how this is in turn related to (un)gauging the categorical SymTFT introduced in section 3.

¹¹Specifically, the identity element in the pre-gauged algebra, $\mathbf{1}$, is the composite algebra \mathcal{A}_C with respect to which the gauging has been performed.

¹²To facilitate the reader, we have added an appendix with further details to be read, albeit we highly recommend to go back to the original paper [34] for a more thorough overview.

¹³More details in this regard will appear in an upcoming work by the same author.

¹⁴For completeness, we provide some further details in appendix A.5.

5.1 Equivariant cohomology, fixed point localisation and convolution product

Equivariant cohomology, [100], also known as Borel cohomology, is a cohomology theory from algebraic topology which applies to topological spaces with a group action. It can be viewed as a generalisation of group cohomology. More explicitly, the equivariant cohomology ring of a space X with action of a topological group G is defined as the ordinary cohomology ring with coefficient ring Λ of the homotopy quotient $EG \times_G X$

$$H_G^\bullet(X; \Lambda) = H^\bullet(EG \times_G X; \Lambda). \quad (5.1)$$

5.1.1 The Kirwan map

The Kirwan map, first introduced in [101], states

$$H_G^\bullet(M) \rightarrow H^\bullet(M//_p G), \quad (5.2)$$

where M is a Hamiltonian G -space, i.e., a symplectic manifold acted on by a Lie group G with a moment map

$$\mu: M \rightarrow \mathfrak{g}^*. \quad (5.3)$$

$H_G^\bullet(M)$ is an equivariant cohomology ring of M , i.e., the cohomology ring of the homotopy quotient $EG \times_G M$ of M by G .

$$M//_p G = \mu^{-1}(p)/G \quad (5.4)$$

is the symplectic quotient of M by G at a regular central value $p \in Z(\mathfrak{g}^*)$ of μ . It is defined as the map of equivariant cohomology induced by the inclusion

$$\mu^{-1}(p) \hookrightarrow M \quad (5.5)$$

followed by the canonical isomorphism

$$H_G^\bullet(\mu^{-1}(p)) = H^\bullet(M//_p G). \quad (5.6)$$

5.1.2 Fixed point localisation and convolution product

In BFN, the embedding of algebras is performed under the so called *convolution product*. To see how this is defined, we start off with the convolution diagram for the affine Grassmannian, [28,29]

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} G_\kappa \times \mathrm{Gr}_G \xrightarrow{q} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G, \quad (5.7)$$

where p, q are projections, m is a multiplication, and

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G = G_\kappa \times_{G_\circ} \mathrm{Gr}_G. \quad (5.8)$$

Hence, given any two G_\circ -equivariant perverse sheaves $\mathcal{A}_1, \mathcal{A}_2$, the pullback

$$p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2) \equiv \mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \quad (5.9)$$

descends to $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ by equivariance, and we can define the convolution product as follows

$$\mathcal{A}_1 \star \mathcal{A}_2 = m_*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2). \quad (5.10)$$

The latter defines a symmetric monoidal structure on the category of $G_{\mathcal{O}}$ -equivariant perverse sheaves on Gr_G , and is equivariant to the monoidal category of finite-dimensional representations of the Langlands dual of G .

(5.8) is the convolution diagram for the case in which G is a complex reductive group and representation $\mathbf{N} = 0$. For arbitrary \mathbf{N} , we can generalise the convolution diagram, achieving a noncommutative algebraic structure. In order to do so, we define a triple $\mathcal{R} = (\mathcal{P}, \varphi, s)$, where \mathcal{P} is a G -bundle on a formal disk $D = \text{Spec } \mathbf{C}[[z]]$, φ is its trivialisation over the punctured disk $D^* = \text{Spec } \mathbf{C}((z))$, and s is a section of the associated vector bundle $\mathcal{P} \times_G \mathbf{N}$ such that it is sent to a regular section of the trivial bundle under φ .

5.1.3 Coulomb branches

Given a complex vector space, V , on a connected reductive algebraic group, G , with a fixed faithful linear action on V , the *Higgs branch* is defined as follows

$$\mathcal{M}_H = \mu^{-1}(0) // G = \text{Spec} \left(\mathbf{C} \left[\mu^{-1}(0) \right]^G \right) \tag{5.11}$$

with

$$\mu : T^*V \rightarrow \mathfrak{g} \tag{5.12}$$

denoting the moment map, whereas the *Coulomb branch*, \mathcal{M}_C , is the spectrum of a ring constructed as a convolution algebra in the homology of the affine Grassmannian, where the choice of the representation V is incorporated as certain quantum corrections to convolution in homology, which are kept track of by an auxiliary vector bundle, [28,29]. \mathcal{M}_H and \mathcal{M}_C are conjectured to be symplectic duals to each other, with the most important relation between them being Koszul duality between the categories of the quantised varieties, [35].

Let G be a simply-connected complex semisimple group, and let G_{ad} be its adjoint form. The group G_{ad} acts on G by conjugation, and G contains a transverse slice Σ for this action, which was first introduced by Steinberg, [105]. The resulting multiplicative universal centraliser is the smooth affine variety

$$Z_G = \left\{ (a, h) \in G_{ad} \times \Sigma \mid a \in G_{ad}^h \right\}. \tag{5.13}$$

When G is simply-laced, Bezrukavnikov, Finkelberg, and Mirković, [100], showed that its coordinate ring is isomorphic to the equivariant K-theory of the affine Grassmannian of the Langlands dual group G^V , therefore, in this case, Z_G is an example of a Coulomb branch as defined by BFN [28,29].

5.2 Moore-Tachikawa varieties

We now turn to the specific case of interest to us, namely Moore-Tachikawa varieties, and their relation with the work of [30,31,43,44]. First and foremost, we will briefly overview how the Higgs and Coulomb branches arise in this setup.

Given a reductive connected group, G , there is a functor defined as follows

$$\eta_{G_C} : \text{Bo}_2 \rightarrow \mathcal{C} \tag{5.14}$$

where \mathcal{C} is a hyperbolic symplectic manifold, whose objects are affine algebraic groups, and whose morphisms

$$\text{Hom}_{\mathcal{C}}(G_1, G_2) \tag{5.15}$$

are isomorphism classes of symplectic¹⁵ varieties with Hamiltonian $G_1 \times G_2$ -action. η_{G_C} is

¹⁵By this we mean affine, Poisson, operationally symplectic, with a Hamiltonian chosen moment map.

fully specified by its action on objects and morphisms, namely

$$\eta_{G_{\mathbf{C}}}(S^1) \mapsto G \quad , \quad \eta_{G_{\mathbf{C}}}(S^2) \mapsto Z_G \quad (5.16)$$

$$\eta_{G_{\mathbf{C}}}(S^1 \times S^1) \mapsto T^*G$$

where Z_G is the *universal centraliser*, consisting of all the pairs (x, g) with $x \in \mathfrak{g}$, \mathfrak{g} Lie, regular, $g \in G$ and $ad(x)_{\mathfrak{g}} = x$. Given $K \subset \mathfrak{g}$ a Kostant slice, and (e, h, f) a principal \mathfrak{sl}_2 -triple in \mathfrak{g} , the Kostant slice can be rewritten as follows

$$K = e + Z(f) \quad (5.17)$$

with $Z(f)$ the centraliser of f in \mathfrak{g} , $\mathfrak{g} \simeq \mathfrak{g}^*$. For a given Kostant slice, the universal centraliser can therefore be redefined as follows

$$Z_G = \{(g, x) | g \in G, x \in K, ad_{\mathfrak{g}}(x) = x\}. \quad (5.18)$$

What we have outlined so far can be suitably adapted to the case of interest to us, namely when Σ is a Riemann surface whose boundary has n components, from which one gets a symplectic variety X with a G^n -action. This symplectic variety, X , is often times referred to as the *Higgs branch*.

5.2.1 The Kostant-Whittaker symplectic reduction

The action of the functor explicited in (5.16), however, is not exhaustive. Indeed, there is one more piece of information that is needed in order for the 2D TFT to be fully determined, and this is its action on the bordism corresponding to the three-punctured sphere.

To see how this relates to quivers, let us consider the case in which X is a symplectic variety with G -action. Let $N \subset G$ be a maximal map, and

$$\chi: N \longrightarrow \mathbf{C} \quad (5.19)$$

a non-degenerate homomorphism such that

$$N \Big/ [N, N] \simeq \bigoplus_{\alpha_i} \mathbf{C}_{\alpha_i}, \quad (5.20)$$

where α_i denote the simple roots. Then, the Kostant-Whittaker reduction on the symplectic variety corresponds to the quotient

$$KW(X) = \mu_N^{-1}(\chi) \Big/ N, \quad (5.21)$$

and the universal centraliser becomes

$$Z_G \simeq KW_{G \times G}(T^*G). \quad (5.22)$$

Furthermore, we can take

$$G \times K = KW_{G^{\vee}}(T^*G) \quad , \quad KW_G(G \times K) = Z_G, \quad (5.23)$$

meaning that filling in a hole on the sphere corresponds to performing a KW_G reduction. Note that this is a statement of 3D mirror symmetry, [93].

According to the Moore-Tachikawa conjecture, overviewed in slightly more detail in appendix A.5, a 2D TFT is an example of a 2-categorical structure, that is the Alday-Gaiotto-Tachikawa dual of a supersymmetric 4D theory arising from dimensional reduction of 6D $\mathcal{N} = (2, 0)$ SCFTs of type A_{N-1} on a sphere with N maximal punctures. These theories, oftentimes denoted $\mathcal{T}_N[\Sigma_{0,k}]$, upon being further dimensionally reduced on a circle, $\mathcal{T}_N[\Sigma_{0,k} \times S^1]$, are known to be related to Sicilian 3D $\mathcal{N} = 4$ theories, the so-called star-shaped quivers, upon taking the $\lim_{S^1 \rightarrow 0}$, [43].

In [103], the authors calculated the Hilbert series of the chiral ring

$$\mathbf{C}[\mathcal{M}_C] \simeq \mathbf{C}[\mathcal{M}_H^{4D}] \tag{5.24}$$

exactly with the intention of obtaining new information about the structure of the Higgs branches of the Higgs branch of the 4D theory, \mathcal{M}_H^{4D} , via a direct analysis of the corresponding Coulomb branches \mathcal{M}_C of $\mathcal{T}_{N,k}$, for general N and k . In addition to this, our work wishes to emphasise the importance of the calculation of the Hilbert series together with the correct identification of the ring elements and relations between them, which plays a crucial role in developing connection with Particle Physics and calculations therein. In achieving the second objective, a crucial role is played by the so-called *abelianisation* procedure, first introduced by [44], and subsequently emphasised also in [43].

5.3 Abelianisation

The basic idea of abelianisation is to embed the Coulomb branch chiral ring $\mathbf{C}[\mathcal{M}_C]$ of a non-abelian theory into a larger, semisimple abelian algebra, \mathcal{A} ,

$$\boxed{\mathbf{C}[\mathcal{M}_C] \hookrightarrow \mathcal{A}}. \tag{5.25}$$

Essentially, abelianisation corresponds to fixed-point localisation in the equivariant homology, which we will briefly explain momentarily. The embedding (5.25) allows to:

1. Verify relations among elements of $\mathbf{C}[\mathcal{M}_C]$.
2. Identify the Poisson structure on $\mathbf{C}[\mathcal{M}_C]$ and its deformation quantisation.
3. Extend the algebra $\mathbf{C}[\mathcal{M}_C]$ over twistor space, ultimately enabling to access the hyperkähler structure on \mathcal{M}_C .

In the context of BFN, this arises when considering a 3D $\mathcal{N} = 4$ gauge theory on $\mathbb{C} \times \mathbb{R}$ spacetime with $\frac{1}{2}$ -BPS boundary conditions \mathcal{B} near spatial infinity on \mathbb{C} .

Working in the cohomology of the Rozansky-Witten theory, \mathcal{Q}_{RW} , the fields may be localised in the QM to sections of bundles that solve the \mathcal{Q}_{RW} BPS equations. This yields the gauged $\mathcal{N} = 4$ QM on

$$\mathcal{M}_{|\mathbb{C}|} = \{ (E, X) \}, \tag{5.26}$$

where E is a holomorphic $G_{\mathbb{C}}$ -bundle on \mathbb{C} , and X is a holomorphic section of an associated R -bundle with gauge group, $calG$ is the holomorphic $G_{\mathbb{C}}$ gauge transformation on \mathbb{C} .

The Hilbert space of the space of the QMs in the \mathcal{Q}_{RW} twist should be the \mathcal{G} -equivariant de Rham cohomology of $\mathcal{M}_{|\mathbb{C}|}$. Turning on the Omega background further, corresponds to

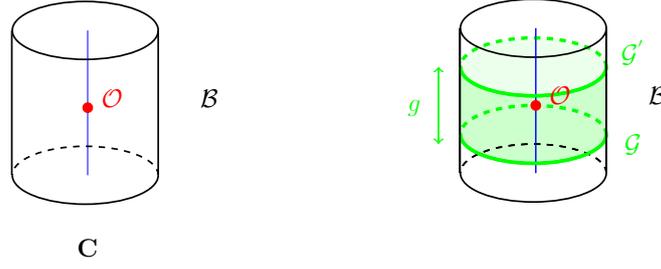


Figure 7: Picture reproduced from the work of [43], encoding the key idea outlined in their treatment, and from which our work was in turn inspired.

working equivariantly w.r.t. the $U(1)_\epsilon$ spatial rotation group of \mathbf{C} , which is an ordinary symmetry of the moduli space, the Hilbert space is

$$\mathcal{H} \stackrel{\text{def.}}{=} H_{\mathfrak{g} \times U(1)_\epsilon}^\bullet(\mathcal{M}_{[\mathbf{C}]}) . \quad (5.27)$$

$\mathcal{M}_{[\mathbf{C}]}$ is contractible to a point where E is the trivial bundle and X is the zero-section. \mathcal{G} is contractible to G , therefore the Hilbert space reduces to

$$\mathcal{H} \simeq H_{\mathfrak{g} \times U(1)_\epsilon}^\bullet(\text{pt.}) = \mathbf{C}[\varphi, \epsilon]^W , \quad (5.28)$$

namely Weyl-invariant polynomials in the equivariant weights $\varphi \in \mathfrak{t}_{\mathbf{C}}$ and ϵ .

We can therefore write the BFN construction, which essentially reads as follows

$$\mathbf{C}_\epsilon[\mathcal{M}_{\mathbf{C}}] \equiv H_{\mathfrak{g} \times \mathfrak{g}' \times U(1)_\epsilon}^\bullet(\mathcal{M}_{[\mathbf{C} \cup \mathbf{C}]}) \simeq H_{\mathfrak{G} \times U(1)_\epsilon}^\bullet\left(\mathcal{M}_{[\mathbf{C} \cup \mathbf{C}]} \Big/ \mathcal{G}'\right) , \quad (5.29)$$

where $\mathcal{G}, \mathcal{G}'$ are the groups of regular holomorphic gauge transformations on the top and bottom copies of \mathbf{C} , and $U(1)_\epsilon$ is the usual spatial rotation group. Importantly, this space, (5.29), is non-contractible, and has highly nontrivial topology since it contains monopole operators. OPEs of local operators in the algebra $\mathbf{C}_\epsilon[\mathcal{M}_\epsilon]$ naturally correspond to the convolution product rather than the cup product.

We are now ready to explain a particularly useful tool in the context of equivariant cohomology, namely fixed-point localisation. Letting $T \subset G$ denote the maximal torus, one finds that $T \times U(1)_\epsilon$ fixed points of $\mathcal{M}_{[\mathbf{C} \cup \mathbf{C}]} \Big/ \mathcal{G}'$ are isolated points described as pairs $(E, X), (E', X')$ where $X = X' = 0$ are zero sections, E is trivial, and E' is obtained from E by a gauge transformation

$$g(z) = z^A \quad , \quad A \in \text{cochar}(G) , \quad (5.30)$$

meaning fixed points are labelled by cocharacters corresponding to abelian monopole operators.

If \mathcal{F} is the fixed point set of the $T \times U(1)_\epsilon$ action on $\mathcal{M}_{[\mathbf{C} \cup \mathbf{C}]} \Big/ \mathcal{G}'$, $\mathcal{F} \simeq \text{cochar}(G)$ is isomorphic to the cocharacter lattice.

The equivariant cohomology of the fixed point set contains a copy of

$$H_{T \times U(1)_\epsilon}^\bullet(\text{pt.}) = \mathbf{C}[\varphi, \epsilon] \quad (5.31)$$

for every point in \mathcal{F} , i.e.,

$$H_{T \times U(1)_\epsilon}^\bullet(\mathcal{F}) \simeq \mathbf{C}[\varphi, \epsilon, \{v_A\}_{A \in \text{cochar}(G)}]. \quad (5.32)$$

In its localised version, one gets that

$$\mathcal{A}_\epsilon \subset H_{T \times U(1)_\epsilon}^\bullet(\mathcal{F})^{\text{loc}}. \quad (5.33)$$

The crucial difference between the two sides in (5.33) is that the localised cohomology indiscriminately inverts all weights, whereas in \mathcal{A}_ϵ we only invert part of them. The localisation theorem provides the following algebraic embedding

$$\mathbf{C}_\epsilon[\mathcal{M}_{[\mathbf{C}]}] = H_{G \times U(1)_\epsilon}^\bullet\left(\mathcal{M}_{[\mathbf{C} \cup \mathbf{C}]} / \mathcal{G}'\right) \hookrightarrow H_{T \times U(1)_\epsilon}^\bullet(\mathcal{F})^{\text{loc}}, \quad (5.34)$$

which is the abelianisation we introduced in (5.25). Importantly, the RHS of (5.34) is the mother algebra we were looking for, the un-gauged algebra referred to in section 3, and the key entity encoding essential information about the QFT in question. In conclusion to this section, we wish to highlight that, for any successful QFT to be fully specified, one should really be able to determine such an embedding. Any shortcomings in achieving it, should be taken as signalling a yet incomplete understanding of the underlying mathematical structure of the theory in question.

6 Where to next?

With the discussion in section 3, we have reached the bifurcation point in diagram 2, and showed the importance of abelianisation in geometric representation theory. Specifically, we highlighted the common ground between [30,31,43].

As explained at the very beginning of our work, one of our key motivations is the need to deepen the mathematical toolbox adopted by Phenomenologists in answering open questions in Particle Physics. Recent developments in the Pheno community have started relying upon the calculation of the Hilbert series, most importantly [46–48], successfully encoding key fundamental processes, such as the Higgs mechanism. The crucial point of our analysis, though, is that, albeit Hilbert series calculations are indeed very useful for keeping track of the degrees of freedom and symmetries of a given theory, in principle they are not enough to determine the exact operator content of the theory in question.

On the other hand, ring homologies, on which Hilbert series are calculated, encode crucial properties of the QFT at hand, enabling a much deeper understanding of its underlying categorical structure. The key take-home message of this work is the crucial role played by abelianisation for identifying the underlying mathematical structure of a given QFT. This concept will be investigated further for the specific case of the SM and its extensions in future work.

Additional complementary directions addressed by the same author will be developing ring homologies for Moore-Tachikawa varieties beyond categorical duality¹⁶, and understanding their implications for Koszul duality. We plan to report on any advancements in this regard in the near future.

¹⁶Thereby following up from the analysis carried out in [36].

7 Conclusions and Outlook

This work is meant to be the first of a series of papers by the same author, continuing from past developments, bridging the gap between the Pure Mathematics and the Theoretical Particle Physics. The purpose of this introductory article was mainly:

- To illustrate (some) key fundamental problems in Theoretical Particle Physics necessitating further mathematical understanding.
- Opening the scene to well-established techniques in representation theory and categorical algebraic geometry to help in this regard.
- For those who are not experts in the field, serves as an oversimplified explanation of the powerful formalism put forward by BFN, and connecting it with Moore-Tachikawa varieties.
- Motivating Phenomenologists to engage more with Pure Mathematicians.

The main objective of this paper was that of explaining the higher categorical structures that are needed for describing the invariants associated to specific supersymmetric quiver gauge theories, with a particular focus on dualities and their mutual relations in terms of higher-categories.

The article was structured as follows:

1. Section 2 explained the motivations and outline of the present and forthcoming work by the same author. In doing so, we briefly introduced scattering amplitudes as crucial tools within the context of particle physics, explaining their connection with Hilbert series. We then briefly explained some of the key open questions in Theoretical Particle Physics necessitating further mathematical understanding, specifically the hierarchy problem, and the lightness of the Higgs mass (and its relation to Supersymmetry (SUSY)).
2. Section 3 is mostly a revision of [30,42], and we used it at this stage for multiple reasons. Primarily because it allows to introduce gauge theories and Lagrangians in a mathematical language¹⁷, which in turn is suitable to make connection with the higher-categorical Symmetry Topological Field Theory (SymTFT) prescription put forward by Freed, Moore, and Teleman, (FMT), [30], in the context of relative QFTs, arising from topological holography.
3. Section 5 was ultimately devoted to explaining in a simplified way the well-established BFN construction leading to ring homologies. The theories in question are 3D $\mathcal{N} = 4$ supersymmetric quiver gauge theories. This section highlights the importance that categorical algebraic geometry plays in identifying ring homologies accounting for the complete spectrum of a given theory, therefore giving a complementary insight with respect to the Hilbert series alone. In so doing, we mostly highlighted the importance of *abelianisation*. In doing so we explicitly showed the relation between the work of [44], that of Freed, Moore, and Teleman (FMT), [30], and Teleman's, [31], which, to the best of our knowledge, has only previously been mentioned in [32,33].

¹⁷Both of which are essential in the description of the SM and its extensions.

4. From the discussion of the previous section, the concluding section, 6, paves the way to forthcoming work, where a more detailed analysis with regard to the algebraic structure and Hilbert series of (Beyond) the Standard Model scenarios will be carried out. Complementarily to this, our objective is that of developing ring homologies for Moore-Tachikawa varieties beyond categorical duality, [36], and understanding their implications for Koszul duality. We plan to report on any advancements in this regard in the near future.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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Appendix A. Algebraic varieties: some more background details

A.1 Riemann surfaces

The geodesic structure of a Riemannian manifold is described by the Hamilton-Jacobi equations. As previously mentioned, the M and Φ defining the sigma-model can both be taken to be Riemannian.

The cotangent bundle $T^*\Phi$ can always be locally trivialised, namely

$$T^*\Phi \Big|_U \simeq U \times \mathbf{R}^n. \quad (\text{A.1})$$

Given g_{ij} on Φ , the Hamiltonian function is defined as follows

$$H(q, p) \stackrel{def.}{=} \frac{1}{2} g^{ij}(q) p_i p_j, \quad (\text{A.2})$$

from which the Hamilton-Jacobi equations define the geodesic (or Hamiltonian) flow on Φ

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \quad (\text{A.3})$$

The Hamiltonian description therefore enables us to interpret the sigma-model as the gluing of two energy functionals, namely the momenta in $T^*\Phi$ and T^*M .

A.2 The sigma-model

By definition, [73], a sigma-model is a field theory describing the field as a point particle confined to move on a fixed manifold. The latter could either be a Riemannian manifold, a Lie group, or a symmetric space.

When coupling it to a gauge field, we get the Landau-Ginzburg (LG)-model.

The Lagrangian density for the sigma-model can be expressed in many different ways. One of the simplest expressions is the following

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n g_{ij}(\phi) \partial^\mu \phi_i \partial_\mu \phi_j, \quad (\text{A.4})$$

where $g_{ij}(\phi)$ denotes the metric tensor on the field space $\phi \in \Phi$, where Φ can be any Riemannian manifold.

In a more fully-geometric notation, it can be written as a fiber bundle with fibers Φ over a differentiable manifold M . Given a section

$$\phi: M \rightarrow \Phi, \quad (\text{A.5})$$

fix a point $x \in M$. Then, the pushforward at x is a map of tangent bundles

$$d_x \phi: T_x M \rightarrow T_{\phi(x)} \Phi, \quad (\text{A.6})$$

taking

$$\partial_\mu \mapsto \frac{\partial \phi^i}{\partial x^\mu} \partial_i. \quad (\text{A.7})$$

The sigma model action then reduces to

$$S = \frac{1}{2} \int_M d\phi \wedge \star d\phi. \quad (\text{A.8})$$

Among the most important interpretation of the classical sigma-model is that of non-interacting QM. Taking $\Phi \equiv \mathbf{C}$, (A.8) becomes

$$S = \frac{1}{2} \langle\langle \phi, \Delta \phi \rangle\rangle, \quad (\text{A.9})$$

with

$$\phi: M \rightarrow \mathbf{C} \quad (\text{A.10})$$

interpreted as a wavefunction, whose Laplacian corresponds to its associated kinetic energy. Hence, the classical sigma-model on \mathbf{C} can be interpreted as the QM of a free particle.

A.3 Moduli spaces

In algebraic geometry, a moduli space is a geometric space (a scheme or an algebraic stack) whose points represent algebro-geometric objects of some kind, or their isomorphic classes. The main motivation for introducing moduli spaces is that of finding solutions of geometric problems.

A scheme is a mathematical structure that enlarges the notion of an algebraic variety, accounting for multiplicities, and allowing varieties defined on any commutative ring. Scheme theory was introduced by A. Grothendieck in 1960 for addressing deep problems in algebraic geometry.

A scheme is a topological space together with commutative rings for all of its open sets, which arises from gluing together spectra of commutative rings along their open subsets, i.e., ringed space which is locally a spectrum of a commutative ring.

Algebraic surfaces can be studied making use of morphisms of schemes. In many cases, the family of all varieties of a given type can itself be viewed as a variety of schemes, known as moduli space.

A.3.1 The nonlinear sigma-model

Considering Minkowski spacetime, and denoting with \mathcal{F} the space of fields, L the Lagrangian, and γ the variational 1-form, \mathcal{M} the space of classical solutions, and ω the local symplectic form. In its classical formulation, the Lagrangian encodes a classical Hamiltonian system once a specified time direction has been selected, thereby breaking Poincaré invariance.

Upon taking a real-valued scalar field

$$\phi : M^n \rightarrow \mathbb{R}. \tag{A.11}$$

The kinetic term in the Lagrangian would therefore read

$$\begin{aligned} L_{kin} &= \frac{1}{2} |d\phi|^2 \Big| d^n x \Big| \\ &= \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \Big| d^n x \Big| \\ &= \frac{1}{2} \left[(\partial_0 \phi)^2 - \sum_{i=1}^{n-1} (\partial_i \phi)^2 \right] \Big| d^n x \Big|. \end{aligned} \tag{A.12}$$

Adding a potential term of degree ≤ 2 of the kind

$$V : \mathbb{R} \rightarrow \mathbb{R}, \tag{A.13}$$

to (A.12), and assuming V is bounded below by choosing suitable coefficients of the polynomial, then one can eliminate the linear and constant terms by assuming the minimum of V is at the origin with $V = 0$ there. The total Lagrangian therefore becomes

$$L = \left(\frac{1}{2} |d\phi|^2 - \frac{m^2}{2} \phi^2 \right) \Big| d^n x \Big|. \tag{A.14}$$

In non-free theories, the potential is no longer quadratic. The simplest example of an interacting theory is the case of the quartic potential. In either case, in a pure mathematical formulation, the non-linear σ -model is characterised by the following data:

1. A Riemannian manifold, X .

2. A potential energy function

$$V : X \longrightarrow \mathbb{R}. \quad (\text{A.15})$$

In this formulation, the space of fields is the mapping space

$$\mathcal{F} = \left(\text{Map}(M^n, X) \right). \quad (\text{A.16})$$

The total Lagrangian therefore reads

$$L = \left[\frac{1}{2} |d\phi|^2 - \phi^* V \right] |d^n x|, \quad (\text{A.17})$$

from which the energy density reads

$$\mathcal{E}(\phi) = \left[\frac{1}{2} |\partial_0 \phi|^2 + \sum_{i=1}^{n-1} \frac{1}{2} |\partial_i \phi|^2 + V \right] |d^{n-1} x|, \quad (\text{A.18})$$

The space of solutions, namely the moduli space of vacua, \mathcal{M}_{vac} , is

$$\mathcal{M}_{vac} \subset \mathcal{F}\mathcal{E}_N, \quad (\text{A.19})$$

with $\mathcal{F}\mathcal{E}_N$ denoting the space of static fields of finite energy.

For the case of a real scalar field with potential V , one usually assumes that the minimum value is at $V = 0$, and therefore that the solutions lie at

$$\mathcal{M} \stackrel{\text{def.}}{=} V^{-1}(0). \quad (\text{A.20})$$

For example, if $X \equiv \mathbb{R}$, and $V = 0$, the resulting theory is that of a massless real scalar field. For this case, one finds that $\mathcal{M}_{vac} \simeq \mathbb{R}$. On the other hand, for the case of a scalar field with potential

$$V(\phi) = \frac{m^2}{2} \phi^2, \quad (\text{A.21})$$

\mathcal{M}_{vac} is a single point $\phi = 0$. For a quartic potential, instead, \mathcal{M}_{vac} is constituted by two points, i.e., the values of ϕ where V is minimised.

A.4 Kostant-Whittaker reduction

Let \mathfrak{g} be the Lie algebra of G and fix a regular \mathfrak{sl}_2 -triple $\{e, \mathfrak{h}, f\}$. The regular nilpotent element e is contained in a unique Borel subalgebra \mathfrak{b} , with corresponding subgroup U .

Assuming we start from a Poisson manifold M with Hamiltonian G -action, the associated moment map reads

$$\mu : M \rightarrow \mathfrak{g}. \quad (\text{A.22})$$

The Whittaker reduction of M is the Poisson manifold

$$\mu^{-1}(\mathfrak{f} + \mathfrak{b}) / U, \quad (\text{A.23})$$

obtained by Hamiltonian reduction with respect to the action of U at a point corresponding to the regular character.

On the other hand, the principal slice of regular elements is

$$S \stackrel{\text{def.}}{=} \mathfrak{f} + \mathfrak{g}^e \subset \mathfrak{g}, \tag{A.24}$$

as defined by Kostant. By definition, S is contained in the space $\mathfrak{f} + \mathfrak{b}$, therefore we obtain

$$\mu^{-1}(S) \simeq \mu^{-1}(\mathfrak{f} + \mathfrak{b}) / U, \tag{A.25}$$

leading to a canonical relative compactification of the universal centraliser Z_G , resulting in a symplectic variety obtained by Whittaker reduction relative to the $G \times G$ -action on the cotangent bundle T^*G_{ad} .

A.5 Moore-Tachikawa varieties

This subsection is structured as follows:

1. At first, we briefly overview the source and target categorical structure proposed in [34] assuming duality.
2. We then explain what categorical duality means from an algebraic perspective.
3. We conclude the section indicating the relation between Moore-Tachikawa varieties and Coulomb branches of quiver gauge theories as an interesting realisation of 3D mirror symmetry, [93], and how the categorical generalisation proposed in this work suggests interesting applications to quiver varieties that will be addressed in more detail in Part III.

A.5.1 Categorical structure assuming duality

As already mentioned in the introduction to Part V, to a given class \mathcal{S} theory, one can assign a 2D TFT valued in a symmetric monoidal category, [34],

$$\boxed{\eta_{G_C} : \text{Bo}_2 \rightarrow \text{HS}} \tag{A.26}$$

The existence of this 2D TFT relies on the the source and target categories satisfying a certain list of properties, [34]. We will not reproduce all of them in our treatment, and refer the interested reader to the original work of Moore and Tachikawa for a detailed explanation. In this first part of the section, we will only point out some of the crucial assumptions made in their work for reasons that will become clear in the following pages.

A.6 Duality

For the purpose of our work, the crucial assumption made in [34] is the duality structure of the source category Bo_2 . As explained in [34], duality implies that the 2-category Bo_2 is fully specified by its objects, S^1 , and 1-morphisms, namely the bordisms depicted in figure 8. The middle bordism, i.e., the one labelled V , is the identity bordism. One can easily see this by noticing that V is topologically equivalent to a cylinder whose edges are the red circles, i.e., the object of 2-category Bo_2 .

For η_{G_C} to be well defined, the source and target categories are required to satisfy certain sewing relations, [34,106]. This practically means that, compositions between morphisms

should close. In particular, the identity itself can be defined in terms of composite homomorphisms as follows,

$$\boxed{U_{G_c} \circ W_{G_c} \equiv T^* G_c}, \quad (\text{A.27})$$

where $T^* G_c \equiv V_{G_c}$. Indeed, one can easily see that combining the first and third bordisms in figure 8, is topologically equivalent to V .¹⁸

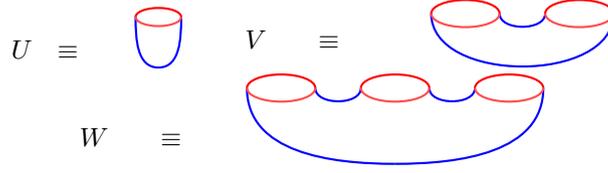


Figure 8: Basic bordisms assuming duality of, both, the source and target categories leading to the definition of the identity element, V_{G_c} , and the maximal dimensional Higgs branch, W_{G_c} .

A.7 Key axiom

(A.27) is essential for us in relating the formalism of [34,106] to the setup of figure 8. In particular, it is what leads to the definition of the triple defining an absolute theory. To see this explicitly, let us recall a crucial axiom required to be satisfied by (A.26), and, therefore, in turn by (A.27), [34].

For $X \in \text{Hom}(G'_c, G_c)$ and $Y \in \text{Hom}(G_c, G''_c)$, their composition

$$Y \circ X \in \text{Hom}(G'_c, G''_c) \quad (\text{A.28})$$

is identified with the holomorphic symplectic quotient

$$\begin{aligned} Y \circ X &\stackrel{\text{def.}}{=} X \times Y // G_c \\ &= \{(x, y) \in X \times Y \mid \mu_X(x) + \mu_Y(y) = 0\} / G_c, \end{aligned} \quad (\text{A.29})$$

where

$$\mu_X : X \longrightarrow \mathfrak{g}_c^* \quad , \quad \mu_Y : Y \longrightarrow \mathfrak{g}_c^* \quad (\text{A.30})$$

are the moment maps of the action of G_c on X and Y , with \mathfrak{g}_c the Lie algebra associated to G_c . The identity element

$$T^* G_c \stackrel{\text{def.}}{=} \text{id}_{G_c} \in \text{Hom}(G_c, G_c) \quad (\text{A.31})$$

comes with a Hamiltonian $G_c \times G_c$ action. As also explained in [34], to see that $T^* G_c$ acts as the identity, it is enough to consider a composition of homomorphisms, $T^* G_c \circ X$. Identifying $T^* G_c \simeq G_c \times \mathfrak{g}_c$, and an element of $T^* G_c$ as (g, a) . The moment map condition, (A.29), reduces to

$$a + \mu(x) = 0, \quad (\text{A.32})$$

¹⁸Indeed, V is topologically equivalent to the cylinder, i.e., the cobordism between S^1 and itself.

from which a can be removed. Consequently, the induced 2-form on the solution space is G_c -invariant and basic. Upon taking the quotient with respect to G_c , we can gauge g to 1, leading to a holomorphic isomorphism with the original X space with its symplectic form.

The categorical quotient taken in defining the composition (A.29) is equivalent to the one that a given absolute theory should be equipped with to potentially gauge away its entire operator content, while leaving only the identity in the spectrum¹⁹. Indeed, this is true as long as the embeddings of the subalgebras associated to G'_c and G''_c are subsets of each other within the mother algebra \mathfrak{g}_c . However, we are interested in describing more general setups, where the embeddings of the algebras are intersecting albeit not one included within the other. In the remainder of our treatment, we will explain that, for this to be described in the formalism of [34,106], the standard identity element associated to the gauge group G_c and embedding Lie algebra \mathfrak{g}_c needs to be removed from Bo_2 , while being replaced by a new composite bordism, and propose the definition of a new functor.

A.8 Algebraic Varieties

In the concluding part of this section, we highlight an interesting relation to 3D mirror symmetry and generalisations thereof, presented by the same author in [32].

The Higgs branches described by Moore and Tachikawa are known to have been reproduced by [28] as the Coulomb branches of 3D $\mathcal{N} = 4$ supersymmetric quiver gauge theories. Such correspondence is therefore equivalent to a statement of 3D mirror symmetry. The purpose of [33] is to explain what the 3D dual of a theory obtained relaxing the categorical duality assumption is in terms of Coulomb branches of 3D $\mathcal{N} = 4$ quiver gauge theories. This enabled us to prove the statements made in [32].

If V has been removed from the source, one should expect there to be more than one 2D TFT of the kind (A.26) associated to two different gauge groups whose embedding in the gauge group associated to the original TFT with identity element V is not simply a cochain complex. Correspondingly, this also means that there is more than one 1-morphism U_{G_c} .

Given that the identity of the embedding theory is defined as follows

$$\boxed{U_{G_c} \circ W_{G_c} \equiv T^*G_c} \quad , \quad \boxed{\eta_{G_c}(V) \equiv T^*G_c} \quad . \quad (\text{A.33})$$

and that

$$U_{G_c} \stackrel{\text{def.}}{=} G_c \times S_n \subset G_c \times \mathfrak{g}_c \simeq T^*G_c, \quad (\text{A.34})$$

with S_n is the Slodowy slice at a principal nilpotent element n . The physical theories of class \mathcal{S} predict the existence of a variety W_{G_c} satisfying the properties needed to define a TFT, η_{G_c} . From the duality assumption, it follows that the dimensionalities of the two varieties are related as follows

$$\dim_c U_{G_c} \stackrel{\text{def.}}{=} \dim_c G_c + \text{rank } G_c. \quad (\text{A.35})$$

$$\dim_c W_{G_c} \stackrel{\text{def.}}{=} 3 \dim_c G_c - \text{rank } G_c. \quad (\text{A.36})$$

However, if the identity needs to be removed from Bo_2 , T^*G_c is not the identity and, in particular (A.36) needs to be redefined precisely because the source is no longer a dual category.

¹⁹cf. explanation in the Introduction. This is basically what leads to the definition of the fiber functor, moment map and Drinfeld center.

As a concluding remark to what we have just said, in [106] they conjecture the following property for the moment maps associated to the G^3 action on the Higgs branch W_{G_c}

$$\mu_i : W_{G_c} \rightarrow \mathfrak{g}_c^* \quad , \quad i = 1, 2, 3. \quad (\text{A.37})$$

This is crucial to our analysis since (A.37) can be inverted to obtain the Higgs branch as a hyperkähler quotient

$$W_{G_c} \equiv \mu^{-1}/G^3. \quad (\text{A.38})$$

However, for the case in which the identity is removed from the source category, (A.38) does not hold anymore precisely because of the lack of permutational symmetry arising in the quotient. In [33] we explained how to define W_{G_c} and its dimensionality for the case involving categories without a duality structure by making use of factorisation homology.

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