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A modified thermodynamics of rotating and charged BTZ black hole

Sudhaker Upadhyay [10] · **Nadeem-ul-islam** [2] · **Prince A. Ganai** [3]

[1] Department of Physics, K.L.S college, Nawada, Bihar-805110, India;
and
Department of Physics, Magadh University, Bodh Gaya, Bihar 824234, India;
and
Visiting Associate, Inter-University Centre for Astronomy and Astrophysics (IUCAA) Pune, Maharashtra-411007, India.
Corresponding author email: sudhakerupadhyay@gmail.com; sudhaker@associates.iucaa.in

[2] Department of Physics, National Institute of Technology, Srinagar, Jammu & Kashmir-190006, India.
email: drnadeemulislam@gmail.com

[3] Department of Physics, National Institute of Technology, Srinagar, Jammu & Kashmir-190006, India.
email: princeganai@nitsri.com

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Abstract. We present the thermodynamics of a charged and rotating BTZ black holes here. In particular, we derive expressions for various macroscopic thermal quantities such as entropy, Hawking temperature, Helmholtz free energy, internal energy, enthalpy, Gibbs free energy, and specific heat. To study the effects of small statistical thermal fluctuations around the equilibrium on thermodynamics, we calculated the leading-order corrections to the various thermodynamical potentials of charged and rotating BTZ black hole and do comparative analysis for the fixed values of charge and angular momentum.

Keywords: BTZ black holes; Thermodynamics; Thermal fluctuation.

1 Overview and motivation

Long ago, it was proposed that a black hole behaves as a thermodynamic object. This hypothesis was based on a logical analogy from the classical theory of general relativity where the square of irreducible mass [1] (or in other words the area of event horizon [2]) never decreases. A similar kind of behavior is replicated in thermodynamics for entropy, i.e., the entropy of any thermodynamical system never decreases in principle. This fundamental behavior of entropy (an increase of entropy) is quoted as the second law of thermodynamics. Thus by analogy, we have the second law of black hole thermodynamics where the role of entropy is assumed to be played by area of the event horizon. This analogy marked the beginning of a new subject that is black hole thermodynamics. For the zeroth and first law of black hole thermodynamics, surface gravity plays the role of the temperature of standard thermodynamics. There exists one to one correspondence between the classical thermodynamics and black hole thermodynamics [3].

Later on, the quantitative relation between the entropy and area of the event horizon was established in 1973 [4, 5]. It is found that entropy and area of event horizon are related through the relation $S = \frac{A}{4}$. It should be emphasized here that the role of pressure in thermodynamics is played by the cosmological constant. Initially, people doubted that the second law of thermodynamics is getting violated due to the assumption that anything that falls into a black hole can never come out. Therefore, there is a loss of entropy (information). To rescue the second law of black hole thermodynamics from being violated these were assigned a maximum entropy [6, 7, 8]. However, the assignment of maximum entropy to black holes led to the discovery of holographic duality [9, 10]. This pictures the degrees of freedom in any region of space-time to its boundary surface and the area-law gets corrected when the size of the black hole is reduced (due to small statistical thermal fluctuations around equilibrium). How does entropy undergo modification by these thermal fluctuations is a very serious question.

In order to resolve this question, people tried different approaches but ended up with the same conclusion that the leading-order corrections to small-sized black holes are logarithmic. For instance, Rademacher expansion of partition was done and resulted logarithmic corrections to the entropy of black hole [11]. The same results were found in the case of string-black hole correspondence too [12]. The correction to entropy due to thermal fluctuations leads to modification in the various thermodynamical equations of states [13, 14, 15, 16, 17, 18, 19, 20]. The effect of small thermal fluctuations on the, BTZ black hole [20] Godel black hole [21] and on the massive black hole in Ads space [22, 23] had been analyzed. Moreover, the first-order leading corrections to the Schwarzchild Beltrami-de-sitter black hole [24] and dilatonic black hole [25] were emphasized. The effect of thermal fluctuations around equilibrium of small-sized black hole was also studied using the partition function approach [26]. The phase structure of the charged Reissener-Nordstram black hole was discussed in [27]. It was observed that they show non-equilibrium second-order phase transition. In [28], Hawking and Page showed that when the temperature of Ads schwarzchild black hole reaches a particular value, called as critical temperature, it undergoes a phase transition. This laid foundation for the study of the P-V criticality of black holes. Later on, a lot of progress has been made and Hawking's work has been extended to other complicated Ads space-times too [29, 30]. Such investigations led to a close analogy between the charged black hole and Van der walls liquid-gas system. Further, in Ref. [31], Faizal et al used an adaptive model of graphene to study some thermodynamic properties of a black hole. The effect of thermal fluctuation on the properties of BTZ black hole in massive gravity has been investigated in Ref. [32]. In Ref. [33], Hawking radiation is studied by using tunneling formalism, and also black hole chemistry for such systems is also analyzed. Moreover, the effect of thermal fluctuation on the thermodynamics of black hole geometry with a hyperscaling violation has been discussed in Ref. [17]. These thermal fluctuations arise from quantum corrections to geometry describing the system under consideration. Corrections to STU black hole have been computed in [34]. These corrections which arise from thermal fluctuations affect the stability of the STU black hole. The effect of leading order corrections to dumb holes (analogs to a black hole), as a consequence of thermal fluctuation, has been studied [35]. The effect of small statistical thermal fluctuations on the entropy of singly spinning Ker-Ads black hole has been investigated in [36]. Furthermore, the corrected thermodynamic equations of states were computed for dilatonic black Saturn [37]. From the perspective of logarithmic corrections to thermodynamics of the modified Hayward black hole, it was found that for such black holes stability is not affected by quantum fluctuations [38]. Recently, emergence of quantum fluctuations to geometry from thermal fluctuations is proposed [39].

Here in this paper, we consider the charged and rotating BTZ black hole as two different thermal systems and discuss their thermodynamics, motivated from entropy-area law. Following a close analogy between the classical thermodynamics and black hole thermodynamics, we first study various thermodynamical equations of states and potentials of the rotating BTZ black holes. For instance, starting from Hawking temperature and entropy for a rotating BTZ metric function, we calculate Helmholtz free energy, internal energy, volume, pressure, enthalpy, and Gibbs free energy of the system in equilibrium. In search of an answer to the question that what happens when small stable fluctuations around the equilibrium of the thermal system are taken into account, we calculate leading-order corrections to the entropy of the rotating BTZ black hole. In order to study the effects of such corrections on the behavior of entropy, we plot the entropy with respect to the event horizon radius for different values of correction parameter and observe that the limiting entropy ($\alpha = 0$) curve at the saddle point is an increasing function and takes only positive values. As expected, the thermal fluctuations affect the entropy of small-sized black holes. We observe two critical values of entropy for black holes where thermal fluctuations do not affect. Between these critical points, entropy has a positive (negative) peak for the positive (negative) correction parameter. Before, the first critical point, corresponding to positive values of correction parameter, the micro-canonical entropy takes a negative asymptotic value which is physically meaningless and forbidden. We also notice that the entropy corresponding to the negative correction parameter takes a positive asymptotic value. Once the Hawking temperature and corrected entropy are known, we derive various corrected thermodynamical potentials of the rotating BTZ black holes to study the effect of thermal fluctuations. In this regard, we start by computing the leading-order corrected enthalpy energy of the system and observe that for small black holes, the enthalpy takes positive (negative) asymptotic value corresponding to positive (negative) correction parameter. We also notice that there exists a critical point. Beyond this critical value, enthalpy increases with the horizon radius. Furthermore, to estimate the possible amount of energy available for doing work, we resort to the derivation of Helmholtz free energy. We observe three critical points for free energy. Two critical points occur in the positive region and one occurs in the negative region. Before the first critical point, opposite to the positive correction parameter, the free energy with a negative correction parameter takes a negative asymptotic value. We then explore the effect of quantum fluctuations on the volume. Once, enthalpy, entropy, temperature, pressure, and volume become known, we estimate the leading order corrections to internal energy and Gibbs free energy. We then scrutinize the effect of thermal fluctuations on the stability of the black hole by studying the nature of corrected specific heat The specific heat becomes positively valued (Corresponding to the negative value of the correction parameter) for small-sized black holes, implying the introduction of stability in the system. A positive value of the correction parameter results in the negative value of specific heat at small horizon radius. After this, we present a brief review of the thermodynamics of non-rotating charged BTZ black hole. The effect of thermal fluctuations on various equations of states of charged BTZ black hole is seen via the derivations of various thermodynamic variables following the same trend as that of uncharged and rotating one. The effect of quantum fluctuations, on the entropy, is similar to that as on the entropy of uncharged rotating BTZ black hole. For the free energy of charged BTZ black hole, we observe two critical points. The first one occurs on the horizon axis while the other one occurs in negative region. Before the first critical point, opposite to the negative correction parameter, the free energy with a negative correction parameter, takes negative asymptotic value. After the first critical point, the correction parameter does not play a significant difference in the free energy. Later on, perturbed enthalpy for charged BTZ black hole is calculated, and it is found that for black holes of the small event horizon, the enthalpy takes negative (positive) asymptotic value for the positive (negative) correction parameter. We also observe the existence of a critical point at a small horizon radius. From the perturbed enthalpy, we derive perturbed volume. We then make the use of corrected enthalpy and volume with pressure as an independent variable, for the derivations of other thermodynamic potentials like internal energy and Gibbs free energy. From the perturbed Gibbs free energy, we observe a critical point beyond which Gibbs free energy remains constant. Before, the critical point, the correction parameter of positive nature, decreases the Gibbs free energy asymptotically. On the contrary, negative valued correction parameter asymptotically increases the Gibbs free energy. Corrected internal energy is then studied and it is found that the a positive value of the correction parameter does not affect the internal energy much and make the least difference by following the same trend as that of uncorrected internal energy curve. On the other hand, the negative value of the correction parameter yields a negative asymptotic value of internal energy. However as the size of the black hole goes on increasing, the difference between the perturbed and unperturbed internal energy gets minimized. Finally, we investigated the stability of charged BTZ black hole and we observed that thermal fluctuations, affect the specific heat in the same fashion as that for rotating one.

This work is presented as follows. In Sec. 2, we discuss the thermodynamics of rotating BTZ black hole and derive the expressions for various equations of states. Within this section, we study the effect of thermal fluctuations on the entropy of rotating BTZ black hole and determine various leading-order corrected thermodynamic variables. The stability of rotating BTZ holes under the effect of thermal corrections is also studied. In Sec. 3, we briefly present the thermodynamics of charged but stationary BTZ black holes and discuss the thermal instability in various thermodynamical variables. We also study the stability of charged BTZ black hole. Finally, in the last section 4, we summarize our results under the heading final remarks.

2 Thermodynamics of rotating BTZ black hole

In this section, we discuss the thermodynamics of the uncharged rotating BTZ black hole. Let us start by writing a metric for the rotating BTZ black hole as follows,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\varphi + g(\varphi)dt^2)^2, \tag{1}$$

where the expressions of metric functions are

$$f(r) = -8GM + \frac{r^2}{l^2} + \frac{16G^2J^2}{r^2}, \tag{2}$$

and

$$g(\varphi) = -\frac{4GJ}{r^2}. \tag{3}$$

From the given lapse function $f(r)$, Eq.(2), it is easy to calculate Hawking temperature T_H as follows,

$$T_H = \left.\frac{f'(r)}{4\pi}\right|_{r=r_+} = \frac{r_+}{2\pi l^2} - \frac{8G^2J^2}{\pi r_+^3}, \tag{4}$$

where r_+ is the horizon radius of black hole obtained by $f(r) = 0$. Using the fundamental postulate of black hole thermodynamics, the corresponding Bekenstein entropy is calculated by

$$S_0 = \frac{\pi r_+}{2G}. \tag{5}$$

This is the same as that of a stationary BTZ black hole. This is because of the fact that the angular moment does not affect the area of the black hole as such entropy of BTZ black hole is not modified by endowing rotations to it.

Now, for a given entropy Eq. (5) and Hawking temperature Eq. (4), we can derive the expression for Helmholtz free energy (denoted by F_r, subscript r, is used to indicate the free energy of rotating BTZ black hole) of rotating BTZ black hole as following:

$$\begin{aligned} F_r &= -\int S_0 dT_H, \\ &= -\int dr_+ \left(\frac{\pi r_+}{2G}\right)\left(\frac{1}{2\pi l^2} + \frac{24G^2J^2}{\pi r_+^4}\right). \end{aligned} \tag{6}$$

Upon solving the above integral, we get

$$F_r = -\frac{{r_+}^2}{8Gl^2} + \frac{6GJ^2}{{r_+}^2}. \tag{7}$$

This is the required expression for the Helmholtz free energy of rotating BTZ black hole. Moreover, it is a matter of calculation to compute the enthalpy energy (H_r) of the system using the following standard formula:

$$H_r = \int T_H dS_0. \tag{8}$$

Plugging the corresponding values of temperature (4) and entropy (5) in the above expression and we have

$$H_r = \frac{{r_+}^2}{8l^2G} + \frac{2GJ^2}{{r_+}^2}. \tag{9}$$

This expression refers to enthalpy energy of rotating BTZ black hole.

The expression for pressure is given by

$$P = \frac{1}{8\pi Gl^2}. \tag{10}$$

Writing H_r, in terms of P, we obtain,

$$H_r = P\pi {r_+}^2 + \frac{2GJ^2}{{r_+}^2}. \tag{11}$$

Hence, the thermodynamic volume is calculated as

$$V_r = \frac{dH_r}{dP} = \pi {r_+}^2. \tag{12}$$

By using the above-derived quantities, we are in state of calculating the further properties of the system such as internal energy (U_r) and Gibbs free energy (G_r). In fact, it is well-known that internal (U_r) is given mathematically by following relation:

$$U_r = H_r - PV_r. \tag{13}$$

Substituting the corresponding values of enthalpy energy, pressure and volume in Eq. (13), we have

$$U_r = \frac{2GJ^2}{{r_+}^2}. \tag{14}$$

This is the expression for internal energy of rotating BTZ black hole. In the same fashion, we would like to derive the expression for Gibbs free energy (G_r). The thermodynamical formula for Gibbs free energy is given by,

$$G_r = F_r + PV_r. \tag{15}$$

For a given values of F_r, P and V_r, the Gibbs free energy is calculated by

$$G_r = \frac{6J^2G}{{r_+}^2}. \tag{16}$$

In the forthcoming sections, we would like to see the consequences of small stable fluctuations around equilibrium on the thermodynamics of rotating BTZ black holes.

2.1 The leading-order perturbed entropy

To calculate the entropy of a thermal system, there are two main ways: (a) by considering a microcanonical ensemble, where an equilibrium system is treated as being genuinely isolated; (b) by considering a canonical ensemble, where the system is treated as being in thermal contact with a very large reservoir at a fixed temperature. For the strongly self-gravitating black holes at equilibrium the canonical ensemble consideration cannot be a good choice. For our case, we follow the latter case as the canonical consideration works well for the small-sized thermal system. We calculate the entropy of rotating BTZ black hole when small stable fluctuations around equilibrium are taken into account. For the canonical black holes, the most convenient starting point is the partition function

$$Z(\beta) = \int_0^\infty dE\rho(E)e^{-\beta E}, \tag{17}$$

where β is the inverse of hawking temperature (for convenience Boltzmann constant is set to unit). Here, density of states $\rho(E)$ can be calculated easily for a given partition function by taking inverse Laplace transform of Eq. 17 as follows

$$\begin{aligned} \rho(E) &= \frac{1}{2\pi i}\int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta Z(\beta)e^{\beta E}, \\ &= \frac{1}{2\pi i}\int_{\beta_0 - i\infty}^{\beta_0 + i\infty} d\beta e^{\ln Z(\beta) + \beta E}.. \end{aligned} \tag{18}$$

Here one should note that the exponential term is nothing but the exact entropy of a black hole, i.e.

$$\mathcal{S}(\beta) = \ln Z(\beta) + \beta E. \tag{19}$$

This exact entropy is the total sum of entropies of all individual subsystems and depends on temperature explicitly. The role of thermal fluctuations is very significant only if the size of the black hole is a very small i.e event horizon radius is very small and this also justifies the canonical consideration of the ensemble. To study the effects of thermal fluctuations on entropy we expand $\mathcal{S}(\beta)$ around equilibrium, via Taylor expansion as following:

$$\mathcal{S}(\beta) = S_0 + \frac{1}{2}(\beta - \beta_0)^2 \left.\frac{d^2\mathcal{S}}{d\beta^2}\right|_{\beta=\beta_0} + \text{(higher-order terms)}, \tag{20}$$

where S_0 indicates the canonical entropy at saddle point equilibrium (temperature). Plugging this expanded value of entropy from Eq. 20 in Eq. 18, we have the following density of states:

$$\rho(E) = \frac{e^{S_0}}{2\pi i}\int d\beta e^{\frac{1}{2}(\beta-\beta_0)^2\frac{d^2\mathcal{S}}{d\beta^2}}. \tag{21}$$

The further simplification of this integral leads to the density of states

$$\rho(E) = \frac{e^{S_0}}{\sqrt{2\pi\frac{d^2\mathcal{S}}{d\beta^2}}}. \tag{22}$$

The logarithm of density of states gives the microcanonical entropy at equilibrium

$$S = \ln\rho = S_0 - \frac{1}{2}\ln\frac{d^2S}{d\beta^2} + \text{ (sub-leading terms)}. \tag{23}$$

This relation holds for all thermodynamic systems (including black holes) considered as a canonical ensemble. The correction term is estimated by [40]

$$\frac{d^2\mathcal{S}}{d\beta^2} = CT_H^2, \tag{24}$$

where C represents the dimensionless specific heat. This leads to the following expression for the microcanonical entropy at equilibrium

$$S = \ln\rho = S_0 - \frac{1}{2}\ln CT_H^2 + \text{ (sub-leading terms)}. \tag{25}$$

In case of BTZ black holes the specific heat coincides with the equilibrium value of canonical entropy at saddle point S_0 as there are no work terms. This suggest the leading-order entropy at equilibrium

$$S = S_0 - \frac{1}{2}\ln S_0T_H^2. \tag{26}$$

Without loss of generality, we replace coefficient $\frac{1}{2}$ in second term by a more general correction parameter "α". This leads to the most general form of corrected entropy

$$S = S_0 - \alpha\ln S_0T_H^2. \tag{27}$$

Here α characterizes the effect of thermal fluctuations on the entropy of system. We observe here that the leading-order correction to canonical entropy of black hole is logarithmic in nature. For our considered system of BTZ black holes, the canonical entropy at saddle point is known from Bekenstein-Hawking area law, from Eq. (5). Substituting the values of Hawking temperature, from (4) and Bekenstein-Hawking area law entropy from (5) in Eq. (27), we get the micro-canonical entropy of rotating BTZ black hole at equilibrium as follows,

$$S_c = \frac{\pi r_+}{2G} - \alpha\ln\frac{({r_+}^4 - 16G^2J^2l^2)^2}{8\pi Gl^2{r_+}^5}. \tag{28}$$

To have a comparative analysis between the corrected and uncorrected entropy at equilibrium, we plot the obtained expression for corrected entropy against the event horizon radius for different values of correction parameter.

From the plot (FIG. 1), we observe that in the limit $\alpha \to 0$, the original entropy curve at saddle point is shown in red color which is an increasing function and takes positive

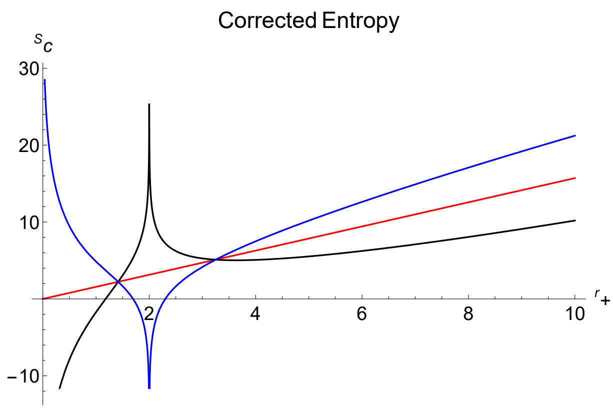

Figure 1: Entropy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

values only. However, entropy shows surprising results when quantum corrections are taken into account for small-sized black holes. However, we notice that the behavior of entropy for large black holes does not undergo any deformation as anticipated. This implies that quantum corrections to entropy are significant only for black holes of very small horizon radius. Also, there exist two critical points. The entropy in between these critical points shows a positive peak for the positive value of the correction parameter, while for negative values of the correction parameter, the corrected entropy shows a negative peak in between the critical points. The negative values of entropy are physically meaningless and therefore forbidden. However, for small black holes whose horizon radii are smaller than the first critical horizon radius, the behavior of corrected entropy is reversed, i.e corresponding to positive values of correction parameter the micro-canonical entropy leads to the negative asymptotic value which is again physically meaningless. The micro-canonical entropy with a negative correction parameter takes a positive asymptotic value and therefore leads to the extra stability in the system. Beyond the second critical point, the micro-canonical entropy in tandem to area-law entropy is an increasing function of event horizon radius irrespective of different specific values of the correction parameter. This justifies the insignificance of thermal fluctuations at a larger event horizon radius.

2.2 The leading-order corrections to thermodynamic potentials

In this section, we would like to evaluate various thermodynamic variables to define the state of the system (ensemble). In this regard, we start with the enthalpy of the system. It is mathematically defined as follows

$$H_c = \int T_H dS_c. \tag{29}$$

where H_c represents corrected enthalpy energy. Plugging the resulting values for Hawking temperature from Eq. (4) and corrected Bekenstein entropy from Eq. (28) in Eq. (29), we obtain

$$H_c = \frac{\alpha 40 G^2 J^2}{3\pi {r_+}^3} + \frac{2GJ^2}{{r_+}^2} - \frac{3\alpha r_+}{2\pi l^2} + \frac{{r_+}^2}{8Gl^2}. \tag{30}$$

This is a leading-order corrected expression for Enthalpy considering thermal fluctuations in account. This leading-order correction terms reflect the effect of thermal fluctuations around the equilibrium. To describe the effect of quantum fluctuations analytically, we plot the resulting enthalpy energy against the event horizon radius for fixed value of angular moment in FIG. 2. Once again from the plot (FIG.2), it is quite clear that thermal fluc-

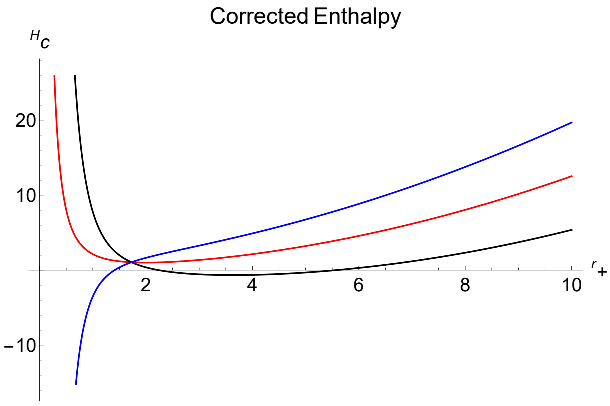

Figure 2: Enthalpy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

tuations are effective only at small event horizon radius while the enthalpy of large sized black holes is unaffected. For black holes of the small event horizon, the enthalpy takes a negative asymptotic value for the negative correction parameter, however, it takes a positive asymptotic value for the positive correction parameter. We also notice that there exists a critical point beyond which internal energy is increasing function of event horizon radius.

Besides enthalpy energy, there are three more important thermodynamic potentials to describe the state of the system, namely Helmholtz free energy, internal energy, and Gibbs free energy. We analyze the effects of thermal fluctuations on these thermodynamic potentials one by one. Being a state function, free energy represents the possible amount of energy available for doing work. Quantitatively, Helmholtz free energy is defined as follows

$$F_c = -\int S_c dT_H. \tag{31}$$

Here F_c denotes corrected free energy. On plugging the values of leading-order corrected Bekenstein entropy and Hawking temperature, the above definition leads

$$\begin{aligned} F_c \;=\; & \frac{\alpha r_+ \log\left(\frac{\left({r_+}^4-16G^2J^2l^2\right)^2}{8\pi G l^4 {r_+}^5}\right)}{2\pi l^2} - \frac{8\alpha G^2J^2\log\left(\frac{\left(r^4-16G^2J^2l^2\right)^2}{8\pi G l^4 r^5}\right)}{\pi r^3} \\ + \; & \frac{40\alpha G^2J^2}{3\pi {r_+}^3} - \frac{3\alpha r_+}{2\pi l^2} + \frac{6GJ^2}{{r_+}^2} - \frac{{r_+}^2}{8Gl^2}. \end{aligned} \tag{32}$$

To analyze the effect of thermal fluctuations on the free energy, we plot the obtained expression (32) with respect to the event horizon radius in FIG. 3. From the plot, we observe three critical points for free energy. Two critical points occur in positive region and one occurs in negative region. Before the first critical point, opposite to the positive correction parameter,

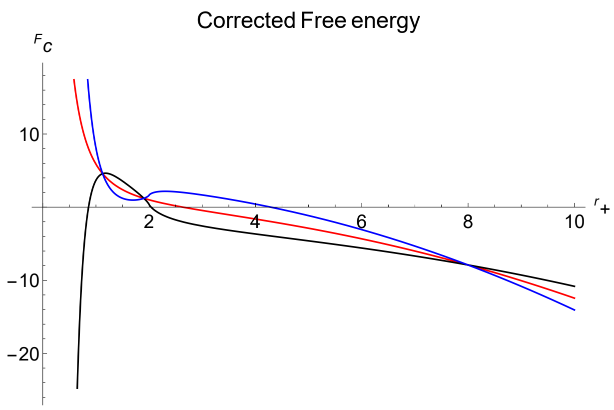

Figure 3: Free energy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

the free energy with negative correction parameter takes negative asymptotic value. After the first critical point, the correction parameter does not play a significant difference for the free energy.

Once we know the corrected expression of corrected enthalpy energy, it is a matter of calculation to find the leading-order corrected volume as following:

$$V_c = \frac{dH_c}{dP}\Big|_{j,r} = \pi {r_+}^2 - 12\alpha G r_+. \tag{33}$$

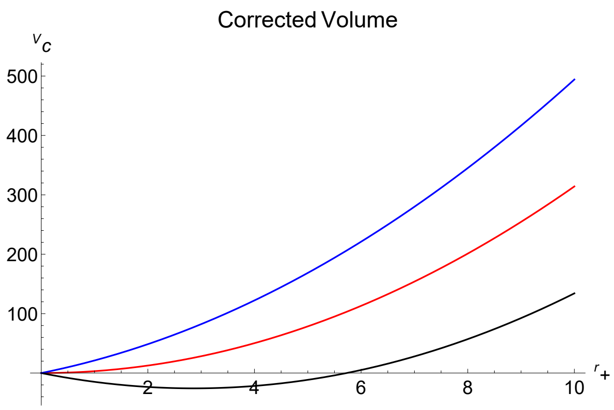

Figure 4: Volume vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Furthermore, we derive the corrected expression for internal energy. Internal energy being a state function has profound significance in thermodynamics and represents the sum total of energy of basic building blocks of a thermodynamic system, i.e., the sum of kinetic energy arising from the motion of particles and potential energy by virtue of the particular configuration of these particles. The corrected internal energy (U_c) can be calculated from the following formula:

$$U_c = H_c - PV_c. \tag{34}$$

Now we have at hand both corrected enthalpy energy and corrected volume. With these quantities we derive the leading order corrections to internal energy as follows,

$$U_c = \frac{\alpha 40 G^2 J^2}{3\pi {r_+}^3} + \frac{2GJ^2}{{r_+}^2}. \tag{35}$$

This represents a quantitative measure of the effect of quantum fluctuations on internal energy. To draw a parallel, between uncorrected and corrected internal energy, the Eq. (35), is plotted against the horizon radius. From the plot, we observe that for small-sized black holes, the uncorrected internal energy is large, however, it goes on decreasing as the size of the black hole increases. The positive value of the correction parameter does not affect the internal energy much and makes the least difference by following the same trend as that of uncorrected internal energy curve. On the other hand, the negative value of the correction parameter yields a negative asymptotic value of internal energy. However as the size of the black hole goes on increasing, the difference between the uncorrected and corrected internal energy gets minimized. It is obvious from the above expression and given plot that in the limit α tends to zero, the uncorrected internal energy is retrieved, as expected.

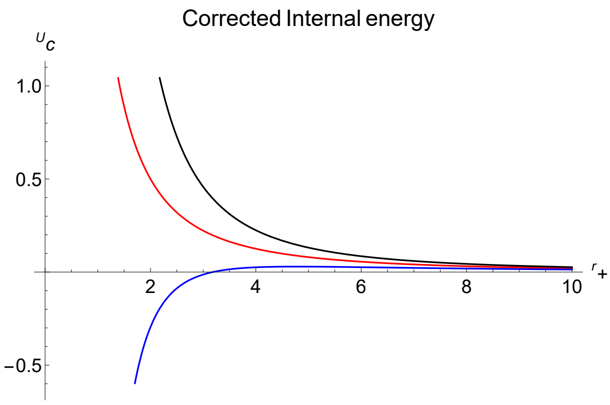

Figure 5: Enthalpy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We will now explore the effects of thermal fluctuations on the fourth thermodynamic potential i.e Gibbs free energy. Gibbs free energy in thermodynamics measures the maximum amount of mechanical work that can be extracted from a system. It is mathematically represented by the relation given below,

$$G_c = F_c + PV_c, \tag{36}$$

where the symbols have their usual meanings Inserting the corrected values of pressure and free energy we obtain,

$$G_c = \frac{80\alpha G^2 J^2 l^2 + 3\alpha\left({r_+}^4 - 16G^2J^2l^2\right)\log\frac{\left({r_+}^4 - 16G^2J^2l^2\right)^2}{8\pi G l^4 {r_+}^5} - 18\alpha {r_+}^4 + 36\pi G J^2 l^2 r_+}{6\pi l^2 {r_+}^3}. \tag{37}$$

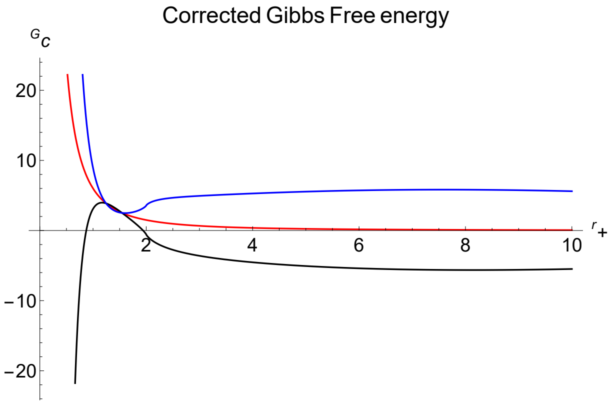

Figure 6: Gibbs free energy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We thus have corrected expression for Gibbs free energy. To appreciate the difference between the corrected and uncorrected Gibbs free energy, we plot the obtained expression, i.e, Eq. (37) against the event horizon radius.

From the plot, we observe that Gibbs free energy undergoes a significant change in its behavior as r_+ tends to zero. We found two critical points at a small event horizon radius. Between these two critical points, thermal fluctuations compel Gibbs free energy to undergo a minuscule but important change. In this region, the positive value of the correction parameter leads to a slight increase in Gibbs free energy, while the negative correction parameter performs the opposite operation in the same region. Before the first critical point, one can examine that the positive (negative) value of the correction parameter produce a negative (positive) asymptotic value of Gibbs free energy. Negativity in Gibbs free energy shows the sign of stability in this region and hints at the maximum amount of energy that can be extracted from the system for useful work

2.3 Stability of rotating BTZ black hole

In order to explore the stability of black holes we study the nature of its specific heat. From the nature of specific heat, we can estimate whether black hole undergoes phase transition or not. The positive values of specific heat confirm that the system is stable against the phase transition, while the negative value of specific heat infers the instability of system. We derive the expression for specific heat by taking thermal fluctuation in account which must reduce to the original expression for uncorrected specific heat when fluctuation is switched-off (i.e. α equal to zero). The specific heat (C_c) can be derived with the help of following well-known formula from classical thermodynamics:

$$C_c = T_H \frac{dS_c}{dT_H}. \tag{38}$$

Inserting the values of the corrected internal energy from Eq. (30) and Hawking temperature from Eq. (4) in the above definition, i.e Eq. (38), we can easily have the expression for leading-order corrected specific heat for the black holes. This reads as

$$C_c = -\frac{160\alpha G^3 J^2 l^2 + 6\alpha G {r_+}^4 + 16\pi G^2 J^2 l^2 r_+ - \pi {r_+}^5}{96 G^3 J^2 l^2 + 2G {r_+}^4}. \tag{39}$$

To analyze the effect of small statistical fluctuations around equilibrium on the stability of our system we plot the corrected specific heat (Eq. 39) against the event horizon radius for fixed values of angular moment. From the plot (FIG. 7), we observe that if there are no

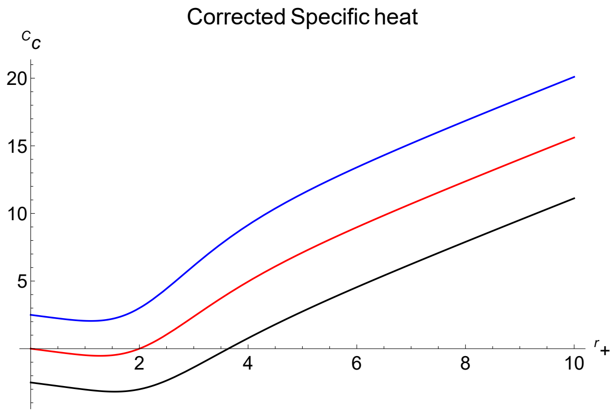

Figure 7: Specific heat vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

thermal fluctuations in consideration (i.e. in the limit α tends to zero), the specific heat becomes less negative-valued for small black hole and takes positive value for large sized black holes. This indicates that without thermal fluctuation, the small sized black holes is unstable and large sized black holes are stable. When thermal fluctuations around the equilibrium are taken into consideration, the specific heat becomes positive valued (Corresponding to negative value of the correction parameter) for small sized black holes also. It implies that thermal fluctuations of such a kind brings stability in the system. Positive value of the correction parameter results in the negative value of specific heat at small horizon radius thereby enhancing the unstable behavior of black hole. For large black holes, thermal fluctuations do not play any significant role which is not surprising.

3 Thermodynamics of charged BTZ black hole

In this section, we first write the metric function characterizing the charged, but non-rotating BTZ black hole as follows

$$f(r) = -8GM + \frac{r^2}{l^2} - \frac{\pi Q^2}{2} \ln \frac{r}{l}. \tag{40}$$

Following the previous section, the Hawking temperature is calculated as follows,

$$T_H = \left. \frac{f'(r)}{4\pi} \right|_{r=r_+} = \frac{r_+}{2\pi l^2} - \frac{Q^2}{8r_+}, \tag{41}$$

where r_+ is the horizon radius of black hole obtained by $f(r) = 0$.

The equilibrium value of entropy is same as that of uncharged one i.e,

$$S_0 = \frac{\pi r_+}{2G}. \tag{42}$$

From the given Hawking temperature and equilibrium entropy, we obtain the free energy for charged BTZ black hole, given by the following expression:

$$F = -\frac{r_+^2}{8Gl^2} - \frac{\pi Q^2 \log \frac{r_+}{l}}{16G} \tag{43}$$

The expression for enthalpy is derived, using the following well-known formula

$$H = \int T_H dS_0. \tag{44}$$

Plugging the corresponding values of temperature from Eq. (41) and entropy from Eq. (42) in the above expression and by solving this, we have

$$H = \frac{{r_+}^2}{8l^2G} - \frac{Q^2\pi}{16G}\ln\frac{r_+}{l}. \tag{45}$$

We can express for enthalpy in terms of pressure as below,

$$H = P\pi {r_+}^2 - \frac{Q^2\pi}{32G}\ln 8\pi GP{r_+}^2. \tag{46}$$

From the Eq. (46), we obtain the expression for thermodynamic volume (conjugate of pressure in thermodynamics) as follows

$$V = \frac{dH}{dP}\Big|_{j,r} = \pi {r_+}^2 - \frac{\pi Q^2}{32GP}. \tag{47}$$

Having expressions for volume, Enthalpy and pressure, we are able to derive internal energy as

$$U = \frac{\pi Q^2}{32G}[1 - \ln 8\pi GP{r_+}^2]. \tag{48}$$

The Gibbs free energy is defined by

$$G = F + PV. \tag{49}$$

Using the corresponding values of free energy, volume and pressure, one gets

$$G = -\frac{Q^2\pi}{32G}(1 + 2\ln\frac{r_+}{l}). \tag{50}$$

This is the equilibrium value of Gibbs free energy for the charged BTZ black hole.

3.1 The corrected thermodynamics of charged BTZ black hole

Thermal fluctuations in thermodynamics mimic the quantum fluctuations in space time geometry. As the black hole evaporates its size decreases in other words it becomes hot as hawking temperature is inversely proportional to mass. Now hot black hole signifies the inevitability of thermal fluctuation and hence quantum fluctuations. These fluctuations modify the thermodynamic behavior of system. Let us look at the corrected version of entropy. To obtain this, we just substitute Eqs. (41) and (42) in Eq. (27), and get the following perturbed entropy of charged BTZ black hole:

$$S_p = \frac{\pi r}{2G} - \alpha \log \frac{\pi r \left(\frac{Q^2}{8r} - \frac{r}{2\pi l^2}\right)^2}{2G}. \tag{51}$$

To have a comparative analysis between the perturbed and exact entropy at equilibrium, we plot the expression for perturbed entropy against the event horizon radius for different values of the correction parameter.

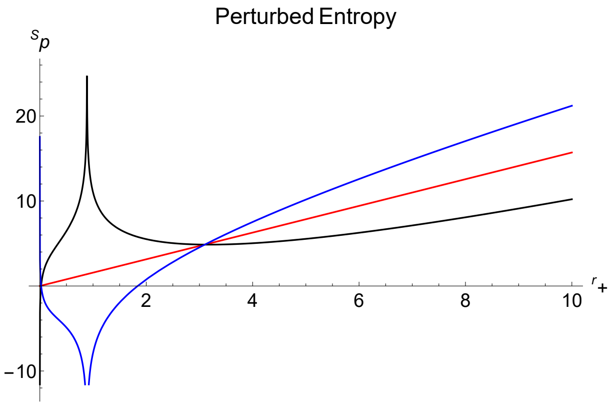

Figure 8: Entropy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = -1.5$ denoted by blue line, $\alpha = -1.5$ denoted by blue line, and $\alpha = 1.5$ denoted by green line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

From the plot (FIG. 8), it is observed that in the limit $\alpha \to 0$ the original entropy curve (as shown in red color) is an increasing function and takes positive values only, as expected. However, entropy shows surprising results when quantum corrections are taken into account for small-sized black holes, in tandem to our assumption while deriving the Generalized form of perturbed entropy i.e Eq. (27). We notice that the behavior of entropy for large black holes does not undergo any deformation as anticipated. This implies that quantum corrections to entropy are significant only for black holes of very small horizon radius. Also, there exist two critical points. The entropy in between these critical points shows a positive peak for the negative value of the correction parameter, while for negative values of the correction parameter, the corrected entropy shows a negative peak in between the critical points. The negative values of entropy are physically meaningless and therefore forbidden. The first critical point happens to be close to the singularity, and before the first critical point, the behavior of thermal fluctuations reverse i.e corresponding to positive values of correction parameter, the micro-canonical entropy leads to the negative asymptotic value which is again physically meaningless. The micro-canonical entropy with negative correction parameter takes a positive asymptotic value and therefore leads to the extra stability in the system. Beyond the second critical point, the micro-canonical entropy in tandem to area-law entropy is an increasing function of event horizon radius irrespective of different specific values of correction parameter. This justifies the insignificance of thermal fluctuations at a larger event horizon radius.

After observing the effect of quantum fluctuations on the entropy, we move on to reckon the perturbed free energy, following the same trend as that for rotating BTZ black hole in the above section, and the derived expression reads,

$$F_p = \frac{1}{8}\left(\frac{\alpha\left(4{r_+}^2-\pi l^2Q^2\right)\log\frac{\pi r_+\left(\frac{Q^2}{8r_+}-\frac{r_+}{2\pi l^2}\right)^2}{2G}}{\pi l^2 r_+}-\frac{12\alpha r_+}{\pi l^2}+\frac{\alpha Q^2}{r_+}-\frac{r_+^2}{Gl^2}-\frac{\pi Q^2\log r_+}{2G}\right) \quad (52)$$

To analyze the effect of thermal fluctuations on the free energy, we plot the obtained

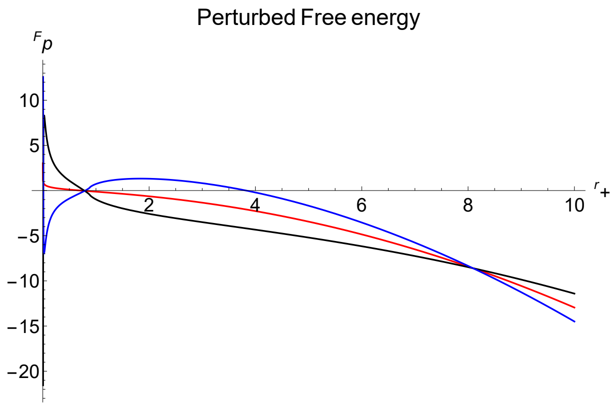

Figure 9: Free energy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

expression (52) with respect to the event horizon radius in FIG. 9.

From the plot (FIG.9), we observe two critical points for free energy. The first one occurs on the horizon axis while as the other one occurs in negative region. Before the first critical point, opposite to the negative correction parameter, the free energy with positive correction parameter takes positive asymptotic value. After the first critical point, the correction parameter does not play a significant difference for the free energy.

Once the entropy and Hawking temperature is known, it is a matter of computation to reckon the perturbed enthalpy. The quantitative expression for perturbed enthalpy is given by,

$$H_p = \frac{1}{16}\left(-\frac{24\alpha r_+}{\pi l^2}+\frac{2\alpha Q^2}{r_+}+\frac{2{r_+}^2}{Gl^2}-\frac{\pi Q^2\log\frac{r_+}{l}}{G}\right). \quad (53)$$

In terms of pressure,

$$H_p = -12\alpha GPr_+ + \frac{\alpha Q^2}{8r} - \frac{\pi Q^2\log 8\pi GPr^2}{32G} + \pi P{r_+}^2. \quad (54)$$

In order to investigate the effect of quantum fluctuations analytically, we plot the resulting enthalpy energy against the event horizon radius for a fixed value of charge in, FIG. 10.

Once again from the plot (FIG.10), it is quite clear that thermal fluctuations are effective only at a small event horizon radius while the enthalpy of large-sized black holes is unaffected. For black holes of small event horizon, the enthalpy takes negative asymptotic value for the negative correction parameter, however it takes positive asymptotic value for positive correction parameter. We also notice that there exists a critical point beyond which perturbed enthalpy energy is increasing function of event horizon radius for negative value of correction parameter. While as it takes negative value from the critical point upto

$r_+ = 6$, and then onwards increses with r_+, for positive values of correction parameter. From the perturbed enthalpy, one can find the expression for perturbed volume simply by

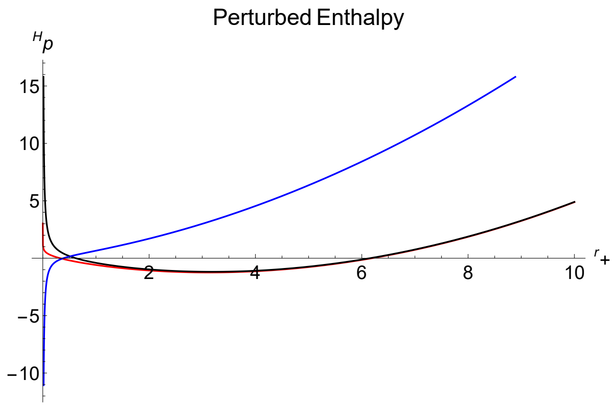

Figure 10: Enthalpy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

differentiating the enthalpy w.r.t to pressure at fixed charge,

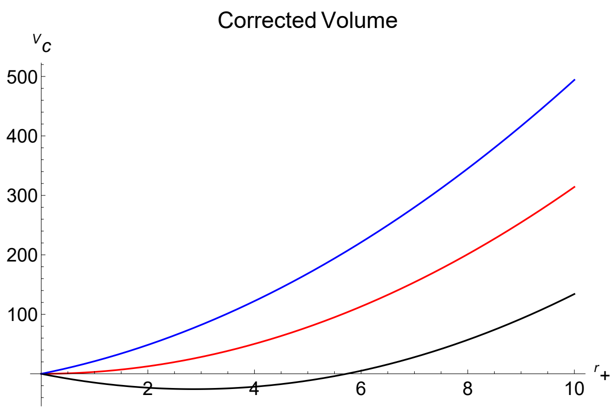

Figure 11: Volume vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$V_p = \frac{dH_p}{dP}\Big|_{j,r} = \pi {r_+}^2 - \frac{\pi Q^2}{32P} - 12\alpha G r_+. \tag{55}$$

The effect of volume is described analytically in plot , FIG. 11. From the same plot, we found that thermal fluctuations measured by negative valued correction parameter, increase the volume and follows the same trend as that of original volume curve, while as the positive valued correction parameter initially decreases the volume up to certain limit, and produces a negative volume region, (which is physically forbidden) and then increases at large distance implying the uselessness of quantum fluctuations at large horizon radius

We now calculate the corrections to Gibbs free energy which reads as follows

$$G_p = -\frac{Q^2\pi}{32G}(1+2\ln r_+) + \frac{\alpha Q^2}{8r_+} + \frac{\alpha\left(4{r_+}^2 + -\pi l^2 Q^2\right)\log\frac{\pi r_+\left(\frac{Q^2}{8r_+}-\frac{r_+}{2\pi l^2}\right)^2}{2G}}{8\pi l^2 r_+}. \tag{56}$$

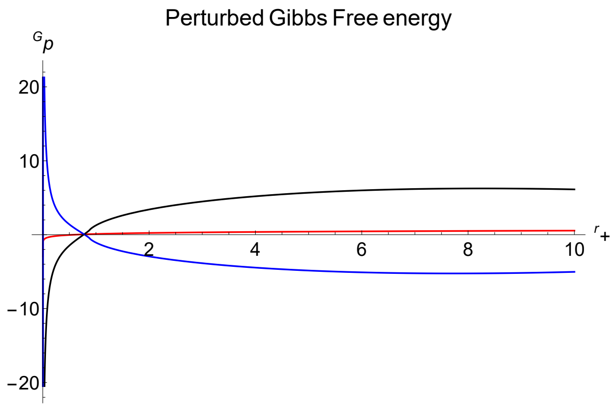

Figure 12: Gibbs free energy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

To analyze the effect of quantum fluctuation and hence thermal fluctuations qualitatively, we resort to graphical line of attack and plot the perturbed Gibbs free energy against the event horizon radius for fixed values of charge and different values of correction parameter as shown in FIG. 12. From the plot (FIG. 12), we discern a critical point beyond which Gibbs free energy remains constant. Before, the critical point, the correction parameter of positive nature, decreases the Gibbs free energy asymptotically. On the contrary, the negative valued correction parameter leads to an asymptotic increase in Gibbs free energy before the critical point.

Another very important thermodynamic potential characterizing the BTZ black hole is internal energy and the perturbed internal energy is given by

$$U_p = \frac{Q^2\pi}{32G}(1-2\log\frac{r_+}{l}) + \frac{\alpha Q^2}{8r_+}. \tag{57}$$

It is obvious from the above expression that in the limit α tends to zero, we recover the uncorrected internal energy as expected. In order to have a qualitative analysis of these corrections, we plot the corrected internal energy against the event horizon radius for various values of correction parameter and observe the consequent effects on the internal energy of the considered system. We notice that at larger event horizon radius the thermal fluctuations fail to affect our system. From the plot, we observe a critical point, beyond the critical point, enthalpy is an decreasing function of event horizon radius irrespective of nature of correction parameter. However, before the critical point, the nature of the correction parameter is of profound importance. In this region, the negative value of the correction parameter α produces a negative asymptotic decrease. On the other hand, the positive value

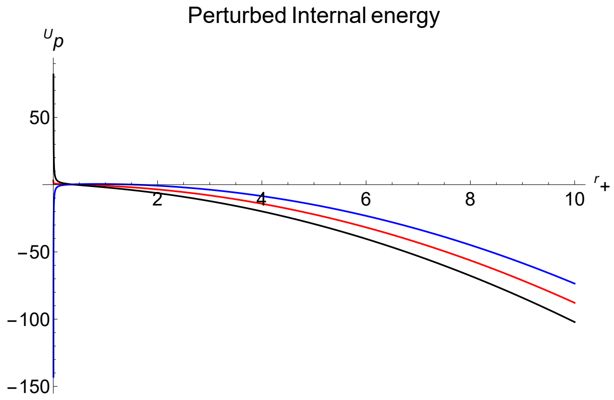

Figure 13: Enthalpy vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of correction parameter produces quite opposite behavior and hence increases the internal energy considerably.

3.2 Stability of charged stationary BTZ black hole

In order to explore the stability of black holes, once again, we resort to stability parameter called specific heat. The specific heat is given by

$$C_c = T_H \frac{dS_p}{dT_H}. \tag{58}$$

Inserting the values of the perturbed entropy from Eq. (51) and Hawking temperature from Eq. (41) in the above definition (58), we can easily have the expression for leading-order perturbed specific heat for the black holes. This reads as

$$C_p = -\frac{2\pi\alpha G l^2 Q^2 + 24\alpha G {r_+}^2 + \pi^2 l^2 Q^2 r_+ - 4\pi {r_+}^3}{2\pi G l^2 Q^2 + 8G {r_+}^2}. \tag{59}$$

To analyze the effect of small statistical fluctuations around equilibrium on the stability of our system we plot the corrected specific heat (59) against the event horizon radius for fixed values of electric charge From the plot (FIG. 14), we found that absence of thermal fluctuations in charged black holes (i.e. in the limit α tends to zero), the specific heat becomes less negative-valued for small black hole and takes positive value for large sized black holes. The perturbed specific heat capacity for charged BTZ follows the similar trend as that of rotating but uncharged BTZ black hole. When thermal fluctuations around the equilibrium are taken into consideration, the specific heat becomes positive valued (corresponding to negative value of the correction parameter) for small sized black holes. It implies that thermal fluctuations of such a kind brings stability in the system in tandem to that of rotating BTZ black hole. Positive value of the correction parameter results in the negative value of specific heat at small horizon radius thereby producing the instability in such black holes.

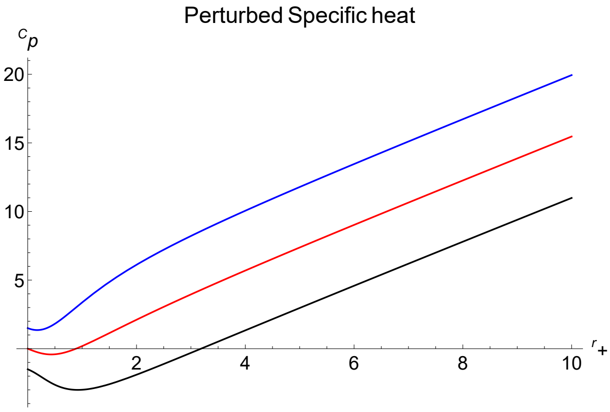

Figure 14: Specific heat vs. the black hole horizon. Here $\alpha = 0$ denoted by red line, $\alpha = 1.5$ denoted by black curve, $\alpha = -1.5$ denoted by blue curve. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

4 Final remarks

Here we have considered the charged and rotating BTZ black holes as two different thermal systems and discussed their thermodynamics, motivated from entropy-area law. Following a close analogy between the classical thermodynamics and black hole thermodynamics, we first studied various thermodynamical equations of states and potentials of the rotating BTZ black holes. For instance, starting from Hawking temperature and entropy for a rotating BTZ metric function, we calculated Helmholtz free energy, internal energy, volume, pressure, enthalpy, and Gibbs free energy of the system in equilibrium. In search of an answer to the question that what happens when small stable fluctuations around the equilibrium of the thermal system are taken into account, we calculated leading-order corrections to the entropy of the rotating BTZ black hole. In order to study the effects of such corrections on the behavior of entropy, we have plotted entropy with respect to the event horizon radius for different values of correction parameter and have found that the limiting entropy ($\alpha = 0$) curve at the saddle point is an increasing function which takes only positive values. As expected, the thermal fluctuations affect the entropy of small-sized black holes. We have found two critical values of entropy for black holes where thermal fluctuations do not affect. Between these critical points, entropy has a positive (negative) peak for the positive (negative) correction parameter. The negative values of entropy are physically meaningless and so forbidden. Before, the first critical point, corresponding to positive values of correction parameter, the micro-canonical entropy takes a negative asymptotic value which is again physically meaningless and forbidden. We have also found that entropy corresponding to the negative correction parameter, takes a positive asymptotic value which, therefore, gives extra stability to the system.

Once the Hawking temperature and corrected entropy became known, we derived various thermodynamical potentials of the rotating BTZ black holes to study the effect of thermal fluctuations. In this regard, we have started by computing the leading-order corrected enthalpy energy of the system and have found that for small black holes the enthalpy energy takes positive (negative) asymptotic value corresponding to positive (negative) correction parameter. We also noticed that there exists a critical point, where the effects of thermal fluctuation are irrelevant. Beyond this critical value, enthalpy energy increases with the horizon radius. Furthermore, to estimate the possible amount of energy available for doing work, we have derived Helmholtz free energy. We observed three critical points in free energy. Two critical points are found in the positive region and one occurred in the negative region. Before the first critical point, opposite to the positive correction parameter, the free energy with negative correction parameter takes negative asymptotic value We then explored the effect of quantum fluctuations on the volume. Once enthalpy, entropy, temperature, pressure, and volume became known, we estimated the leading order corrections to internal energy and Gibbs free energy. We then tried to understand the effect of thermal fluctuations on the stability of the black hole by studying the nature of corrected specific heat The specific heat becomes positively valued (corresponding to negative value of the correction parameter) for small-sized black holes, implying the introduction of stability in the system. A positive value of the correction parameter results in the negative value of specific heat at small horizon radius. After this, we present a brief review of the thermodynamics of non-rotating charged BTZ black hole. The effect of thermal fluctuations on various equations of states of charged BTZ black hole is seen via the derivations of various thermodynamic variables following the same trend as that of uncharged and rotating one. The effect of quantum fluctuations, on the entropy, is similar to that as on the entropy of uncharged rotating BTZ black hole. For free energy of charged BTZ black hole , two critical points are seen to exist. The first one occurs on the horizon axis while the other one occurs in the negative region. Before the first critical point, opposite to the negative correction parameter, the free energy with negative correction parameter, takes negative asymptotic value. After the first critical point, the correction parameter does not play a significant difference in the free energy. Perturbed enthalpy for charged BTZ black hole is calculated, and it is found that for black holes of the small event horizon, the enthalpy takes negative (positive) asymptotic value for the positive (negative) correction parameter. We also observed the existence of a critical point at a small horizon radius. From the perturbed enthalpy, we derived perturbed volume. We then made the use of corrected enthalpy and volume with pressure as an independent variable, for the derivations of other thermodynamic potentials like internal energy and Gibbs free energy. From the perturbed Gibbs free energy, we observed a critical point beyond which Gibbs free energy remained constant. Before, the critical point, the correction parameter of positive nature, decreased the Gibbs free energy asymptotically. On the contrary, the negatively valued correction parameter asymptotically increased the Gibbs free energy. Corrected internal energy is then studied and it was found that

the positive value of the correction parameter did not affect the internal energy much and made the least difference by following the same trend as that of uncorrected internal energy curve. On the other hand, the negative value of the correction parameter yielded a negative asymptotic value of internal energy. However as the size of the black hole went on increasing, the difference between the uncorrected and corrected internal energy got minimized. Finally, we investigated the stability of charged BTZ black hole and we observed that thermal fluctuations, affect the specific heat in the same fashion as that for rotating one. It would be interesting to explore the effect of thermal fluctuations on the $P-V$ criticality of the system by considering the BTZ black holes as a Van der walls fluid system. This is matter of future investigation.

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