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Regular article

Magnetic AdS Black Holes: Topological Thermodynamics and Photon Sphere Analysis

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Abstract. We study the thermodynamic topology of magnetic AdS black holes, with a focus on topological charges and their configurations. Zero points in the system's vector field correspond to topological charges, whose numbers change with key parameters. Across all free parameter ranges, the total topological charge consistently matches that of AdS Reissner-Nordström black holes. Also, we analyze photon spheres, revealing their role in black hole structures and resilience to parameter variations. Unstable photon spheres show distinctive signatures, showing black holes' spacetime geometry. This topological framework offers a systematic approach to understanding black hole stability, phase transitions, and potential astrophysical phenomena.

Keywords: Photon Spheres; Thermodynamic Topology, Magnetic AdS Black Holes.

Contents

1 Introduction

Black hole thermodynamics stands as a cornerstone in the unification of thermodynamics, quantum theory, and general relativity, offering a crucial platform for exploring the microscopic structure of spacetime and the quantum nature of gravity. A prevailing interpretation views the entropy of a black hole as a measure of the statistical degeneracy of its underlying quantum states, primarily concentrated near the event horizon [1]. This concept has driven extensive developments across a wide range of theoretical models and gravitational frameworks [2]. A seminal milestone in this field is the Hawking–Page transition [3], which reveals a thermodynamic phase change between thermal radiation and stable black hole configurations in anti–de Sitter (AdS) spacetimes. This transition is now understood as the gravitational dual of the confinement/deconfinement transition in gauge field theories, a key result within the AdS/CFT correspondence. Furthermore, black hole phase transitions exhibit deep analogies with those of ordinary thermodynamic systems; in particular, the coexistence of small and large black hole phases mirrors the liquid–gas transition governed by the van der Waals equation of state [4].

More recently, topological techniques have emerged as powerful diagnostic tools for investigating the global behavior of black hole phase transitions [5,6]. Among these, Duan's topological current theory provides a unified framework for describing phase structures through conserved topological charges. Building on this foundation, S. W. Wei and collaborators have developed two complementary schemes: the temperature-based (T) approach and the free-energy-based (F) approach [5,6]. The T method treats the temperature as the principal variable, decoupling it from other thermodynamic parameters such as pressure, and employs auxiliary factors (for example, $1/\sin\theta$) to define a temperature field in a two-dimensional parameter space. The zeros of this field correspond to critical points, each associated with a distinct topological charge. Although this approach efficiently locates phase transition points, it provides limited information about the underlying transition mechanism.

In contrast, the F method formulates the analysis in terms of the Helmholtz free energy, interpreting black hole states as topological defects within the thermodynamic landscape. By examining the curvature and winding structure of the free energy surface, this method differentiates between first- and second-order transitions. Multiple minima in the free energy correspond to first-order transitions involving metastable configurations, while smooth, continuous behavior signifies a second-order transition [5–36]. Within this framework, thermodynamic stability is characterized by the sign of the second derivative of the free energy: positive curvature indicates stable phases, while negative curvature corresponds to unstable ones. Together, the T and F methods provide complementary insights—one focusing on temperature evolution, the other on the global topological structure of the free energy surface—offering a comprehensive topological interpretation of black hole phase behavior.

Parallel to thermodynamic investigations, increasing attention has been devoted to the topological study of photon spheres—closed null orbits that confine light in the vicinity of a black hole. These structures play a central role in shaping astrophysical observables such as black hole shadows and gravitational lensing patterns [37–42]. Applying topological principles to photon sphere configurations reveals their intrinsic stability and dynamical characteristics, linking geometric properties with underlying gravitational behavior. Such an approach provides a unified description that bridges topology, spacetime geometry, and the dynamics of null geodesics.

The present work focuses on the nonminimal Einstein–Yang–Mills (NMEYM) theory in an AdS background. By implementing topological methods within both thermodynamic and photon-sphere analyses, we aim to study black hole physics. This perspective not only refines the theoretical understanding of phase structures but also establishes a robust foundation for examining potential observational consequences in strong-gravity regimes.

2 The Model

The NMEYM theory in an AdS spacetime is described by the action [43,44]:

$$S_{\text{NMEYM}} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{8\pi G} - \frac{1}{2}F^{(\alpha)}_{jk}F^{jk(\alpha)} + \frac{1}{2}\mathcal{R}^{jkpn}F^{(\alpha)}_{jk}F^{(\alpha)}_{pn}\right], \tag{2.1}$$

where $g = \det(g_{jk})$ and R represent the determinant of the metric tensor and the Ricci scalar, respectively. The cosmological constant is defined as $\Lambda = -6/l^2$, where l is the AdS radius. The Latin indices (j, k, p, n) range from 0 to 3, while the internal indices (α) run from 1 to 3. The SU(2) Yang–Mills field strength $F^{(\alpha)}_{pn}$ is related to the vector potentials $A^{(\alpha)}_p$ by:

$$F^{(\alpha)}_{pn} = \nabla_p A^{(\alpha)}_n - \nabla_n A^{(\alpha)}_p + f^{(\alpha)}{}_{(b)(c)}A^{(b)}_p A^{(c)}_n, \tag{2.2}$$

where ∇_p denotes the covariant derivative and $f^{(\alpha)}{}_{(b)(c)}$ are the real structure constants of the SU(2) gauge group. The nonminimal susceptibility tensor $\mathcal{R}^{jkpn}$ is defined as:

$$\mathcal{R}^{jkpn} = \frac{q_1}{2}R(g^{jp}g^{kn} - g^{jn}g^{kp}) + \frac{q_2}{2}R\left(R^{jp}g^{kn} - R^{jn}g^{kp} + R^{kn}g^{jp} - R^{kp}g^{jn}\right) + q_3R^{jkpn}, \tag{2.3}$$

where R_{jk} and R_{jkpn} are the Ricci and Riemann tensors, respectively, and q_1, q_2, q_3 are phenomenological parameters describing the nonminimal coupling between the Yang–Mills and gravitational fields. As shown in [43], by setting $q_1 = -\xi$, $q_2 = 4\xi$, and $q_3 = -6\xi$, the equations of motion derived from the above action admit an analytical static spherically symmetric black hole solution:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{2.4}$$

with the metric function

$$f(r) = 1 + \left(\frac{r^4}{r^4 + 2\xi Q_m^2}\right)\left(-\frac{2M}{r} + \frac{Q_m^2}{r^2} + \frac{r^2}{l^2}\right), \tag{2.5}$$

where ξ is the nonminimal coupling parameter and Q_m is the magnetic charge of the Wu–Yang gauge field. The horizons of the black hole are determined by the condition $f(r) = 0$. The two positive roots correspond to the Cauchy horizon (r_-) and the event horizon (r_h). When these two horizons coincide, the black hole becomes extremal. The black hole mass can be expressed in terms of the event horizon radius r_h as:

$$M = \frac{\xi Q_m^2}{r_h^3} + \frac{Q_m^2}{2r_h} + \frac{r_h^3}{2l^2} + \frac{r_h}{2}. \tag{2.6}$$

In the standard geometric approach, the Hawking temperature is determined from the periodicity of the Euclidean time coordinate at the horizon:

$$T = \frac{1}{4\pi}f'(r_h), \tag{2.7}$$

which yields:

$$T = \frac{-6\xi l^2 Q_m^2 + l^2 Q_m^2 r_h^2 - l^2 r_h^4 - 3r_h^6}{8\pi\xi l^2 Q_m^2 r_h + 4\pi l^2 r_h^5}. \tag{2.8}$$

The thermal entropy of the black hole can be obtained from the Wald entropy formula:

$$S = -2\pi \int_\Sigma \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu}\epsilon_{\rho\sigma}\sqrt{h}\, d^2x, \tag{2.9}$$

which leads to

$$S = 4\pi \left(\frac{1}{2\kappa} + \frac{q_1}{2}\frac{\nu^2}{r_h^4}\right) 4\pi r_h^2. \tag{2.10}$$

We finally obtain:

$$S = \pi r_h^2 - 2\pi\xi \frac{Q_m^2}{r_h^2}. \tag{2.11}$$

The first term corresponds to the standard Bekenstein–Hawking area law, while the second term represents the correction due to the nonminimal coupling between the Yang–Mills field and the gravitational background.

3 Topological Structure in Black Hole Thermodynamics

In the study of black hole thermodynamics, one can represent the configuration space as a generalized thermodynamic potential surface. This potential, which serves as a free energy function, is typically formulated in terms of key state variables such as the black hole mass and its temperature. Thermodynamic equilibrium arises when the Euclidean time coordinate τ possesses a periodicity that matches the inverse Hawking temperature T^{-1}, ensuring a self-consistent, on-shell Euclidean manifold [5,6]. To explore the topological properties of this thermodynamic system, an auxiliary vector field $\boldsymbol{\phi}$ is defined over the parameter space. The components of this field correspond to derivatives of the generalized free energy with respect to the relevant parameters. This framework has been successfully employed in numerous gravitational settings, including asymptotically AdS and dS spacetimes, as well as modified gravity theories and systems with non-trivial matter sectors.

The generalized Helmholtz free energy $\mathcal{F}$ can be expressed as [5–36]

$$\mathcal{F} = M - \frac{S}{\tau}, \tag{3.1}$$

where M represents the black hole mass, S is the Bekenstein–Hawking entropy, and τ denotes the Euclidean time period, related to the temperature by $T = \tau^{-1}$. Physical equilibrium configurations occur only when τ satisfies $\tau = T^{-1}$, corresponding to the on-shell regime [5,6].

The associated topological vector field is constructed as

$$\boldsymbol{\phi} = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot\Theta\csc\Theta\right), \tag{3.2}$$

where r_h denotes the horizon radius, and $\Theta \in [0, \pi]$ is an angular coordinate. At the polar boundaries $\Theta = 0, \pi$, the component ϕ_Θ becomes singular, with the field lines extending outward.

Following Duan's ϕ-mapping topological current approach [5,6], the corresponding conserved current is defined by

$$j^{\mu} = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\epsilon^{ab}\partial_{\nu}n^{a}\partial_{\rho}n^{b}, \tag{3.3}$$

where $\mu, \nu, \rho = 0, 1, 2$ denote spacetime indices, and a, b refer to internal field components. The current j^{μ} is localized at the zeros of $\boldsymbol{\phi}$, representing the critical points of the free energy landscape. Integrating over a two-dimensional surface Σ yields the total topological charge:

$$W = \int_{\Sigma} j^{0} d^{2}x = \sum_{i=1}^{n} \zeta_i \eta_i = \sum_{i=1}^{n} \omega_i, \tag{3.4}$$

where ζ_i denotes the Hopf index counting how many times $\boldsymbol{\phi}$ encircles the i-th zero, and $\eta_i = \text{sign}\big[j^{0}(\phi/x)_{z_i}\big]$ is the Brouwer degree, taking values ± 1. Their product $\omega_i = \zeta_i \eta_i$ gives the winding number associated with each zero point. In what follows, we apply this formalism to the thermodynamic topology of the Ayón–Beato–García (ABG) black hole. From Eq. (3.1), the generalized free energy in this geometry takes the explicit form

$$\mathcal{F} = \frac{2\pi\left(4Pr_h^6\tau + 6\xi Q_m^2 r_h - 3r_h^5\right) + 3\tau\left(Q_m^2\left(2\xi + r_h^2\right) + r_h^4\right)}{6r_h^3\tau}, \tag{3.5}$$

and the corresponding vector field components, derived from Eq. (3.2), are

$$\begin{aligned} \phi^{r_h} &= \frac{\tau\left(8\pi Pr_h^6 - Q_m^2\left(6\xi + r_h^2\right) + r_h^4\right) - 4\pi r_h\left(2\xi Q_m^2 + r_h^4\right)}{2r_h^4\tau}, \\ \phi^{\Theta} &= -\frac{\cot\Theta}{\sin\Theta}. \end{aligned} \tag{3.6}$$

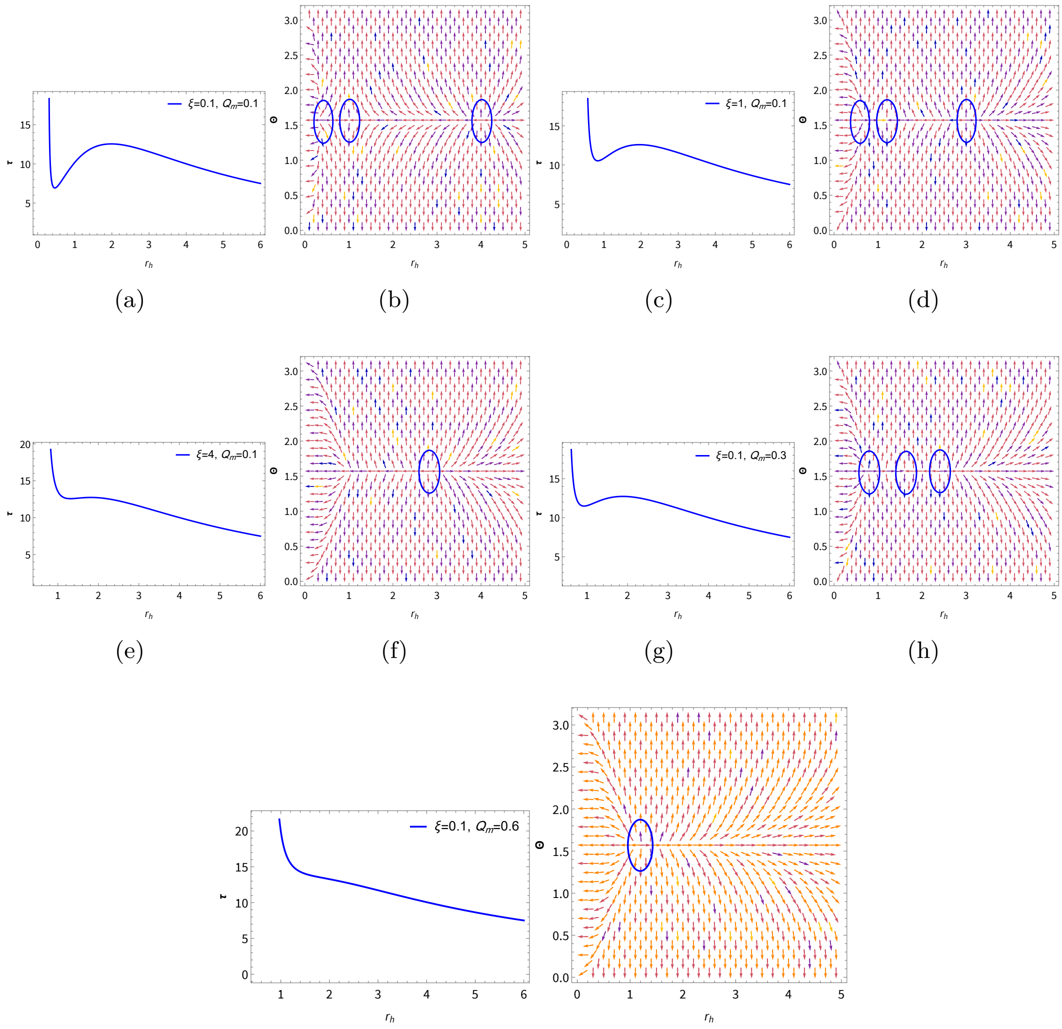
The zeros of $\boldsymbol{\phi}$ determine the equilibrium states, leading to the relation

$$\tau = \frac{4\pi\left(2\xi Q_m^2 r_h + r_h^5\right)}{8\pi Pr_h^6 - 6\xi Q_m^2 - Q_m^2 r_h^2 + r_h^4}. \tag{3.7}$$

We proceed to examine the thermodynamic topology of the magnetic AdS black hole, emphasizing the configuration of topological charges as illustrated in Fig. 1. The normalized vector field depicted in this figure offers a clear depiction of the system's topological arrangement. In particular, Fig. 1 reveals two prominent zero points at specific (r_h, Θ) positions, determined by the parameter sets $\xi = 0.1, 1, 4$ and $Q_m = 0.1, 0.3, 0.6$. These zero points represent topological charges and are enclosed by blue contour loops whose geometry varies with changes in the free parameters.

An important observation is that for small values of ξ and Q_m, the system exhibits three topological charges ($\omega = +1, -1, +1$). As the parameter values increase, only a single topological charge ($\omega = +1$) remains. This classification is robust across the considered parameter space, resulting in a total topological charge of $W = +1$, as demonstrated in Fig. 1. The stability of these configurations is further supported by the evaluation of winding numbers around the zero points.

To formalize this framework, we interpret the free energy as a scalar function over the (r_h, Θ) plane. The associated vector field $\boldsymbol{\phi}$ is constructed so that its zeros correspond to the extrema of the free energy. The rotation of the field around each zero—depending on whether it is a maximum or minimum—allows the assignment of a topological charge to each point [5,6].



(a) (b) (c) (d)

(e) (f) (g) (h)

(i) (j)

Figure 1: The relationship between the Euclidean time period (τ) and the horizon radius (r_h) for a magnetic AdS black hole is illustrated in Figs. 1a, 1c, 1e, 1g, and 1i. These plots reveal how thermodynamic variables evolve with changes in (r_h) and identify critical points associated with phase transitions. Additionally, the orientation of the normalized vector field (n) is mapped onto the (r_h, Θ) plane, indicating the locations of zero points (ZPs) at specific (r_h, Θ) coordinates. The ZPs are observed for parameter values ($\xi = 0.1, 1, 4$) while the magnetic charge is held fixed at ($Q_m = 0.1, 0.3, 0.6$). The distribution of these zero points is fundamental for analyzing both the topological structure and stability behavior of the black hole's thermodynamic landscape.

Classical black hole solutions exhibit characteristic topological charges, such as the Schwarzschild black hole (W = -1), the Reissner–Nordström black hole (W = 0), and the AdS Reissner–Nordström black hole (W = +1) [5,6]

These models serve as benchmarks for categorizing black hole thermodynamics and topological behavior. Since configurations with $W = +1$ correspond to the AdS Reissner-Nordström class, our findings confirm that, across different parameter regimes, the magnetic AdS black hole consistently aligns with this category. Specifically, the observed total topological charge $W = +1$ is in agreement with the established properties of AdS Reissner-Nordström black holes [5,6]. This topological analysis not only corroborates theoretical expectations but also provides a systematic tool for investigating black hole stability and phase transitions, offering insights relevant to both gravitational thermodynamics and potential astrophysical signatures.

4 Photon Spheres

To explore the photon sphere in black hole backgrounds. Photon motion is governed by null geodesics, and the radial dynamics can be described by an effective potential method. Owing to the inherent $\mathbb{Z}_2$ reflection symmetry, it suffices to restrict the motion to the equatorial plane $\theta = \pi/2$ without loss of generality. The radial equation for null trajectories reads

$$\dot{r}^2 + V_{\text{eff}}(r) = 0, \tag{4.1}$$

with the effective potential given by

$$V_{\text{eff}} = g(r)\left(\frac{L^2}{r^2} - \frac{E_p^2}{f(r)}\right), \tag{4.2}$$

where E_p and L denote the photon's conserved energy and angular momentum, associated with the timelike and axial Killing vectors ∂_t and ∂_ϕ, respectively. The functions $f(r)$ and $g(r)$ characterize the spacetime geometry. The photon sphere radius r_{ps} is determined by the critical conditions

$$V_{\text{eff}} = 0, \quad \frac{dV_{\text{eff}}}{dr} = 0, \tag{4.3}$$

leading to the compact expression

$$\left(r^{-2} f(r)\right)'\Big|_{r=r_{\text{ps}}} = 0. \tag{4.4}$$

Here, a prime denotes differentiation with respect to r. The stability of the photon sphere is assessed from the sign of the second derivative of V_{eff}. Specifically,

$$\frac{d^2 V_{\text{eff}}}{dr^2} < 0,$$

corresponds to an unstable photon sphere, while

$$\frac{d^2 V_{\text{eff}}}{dr^2} > 0,$$

indicates stability. Upon further differentiation, the governing relation becomes

$$2rf(r) - r^2 f'(r) = 0, \tag{4.5}$$

which fixes the photon sphere's radial position. At the event horizon $r = r_h$, satisfying $f(r_h) = 0$, the first term of Eq. (4.5) vanishes. Unless $f'(r_h) = 0$—the extremal case—the condition cannot hold at the horizon. Thus, in general, $r_{\text{ps}} \neq r_h$. For extremal black holes, where $f(r_h) = f'(r_h) = 0$, the photon sphere coincides with the degenerate horizon. Beyond the classical potential analysis, one may associate a topological invariant with each photon sphere, providing a global characterization of its stability properties. To this end, consider the scalar function

$$H(r,\theta) = \sqrt{\frac{-g_{tt}}{g_{\phi\phi}}} = \frac{1}{\sin\theta}\sqrt{\frac{f(r)}{h(r)}}, \tag{4.6}$$

where $g_{tt} = -f(r)$ and $g_{\phi\phi} = h(r)\sin^2\theta$ are metric components. The photon sphere satisfies

$$\frac{\partial H}{\partial r} = 0. \tag{4.7}$$

Introducing the vector field $\boldsymbol{\varphi} = (\varphi^r, \varphi^\theta)$, we define its components as normalized gradients of H:

$$\varphi^r = \sqrt{g(r)}\frac{\partial H}{\partial r}, \quad \varphi^\theta = \frac{1}{r}\frac{\partial H}{\partial \theta}. \tag{4.8}$$

The field can be rewritten in polar representation:

$$\boldsymbol{\varphi} = |\varphi| e^{i\Theta}, \quad \text{with} \quad |\varphi| = \sqrt{\varphi^r\varphi_r + \varphi^\theta\varphi_\theta}. \tag{4.9}$$

Equivalently, in complex notation,

$$\varphi = \varphi^r + i\varphi^\theta. \tag{4.10}$$

The normalized components read

$$n^a = \frac{\varphi^a}{|\varphi|}, \quad a = 1,2, \quad \text{with} \quad \varphi^1 = \varphi^r,\ \varphi^2 = \varphi^\theta. \tag{4.11}$$

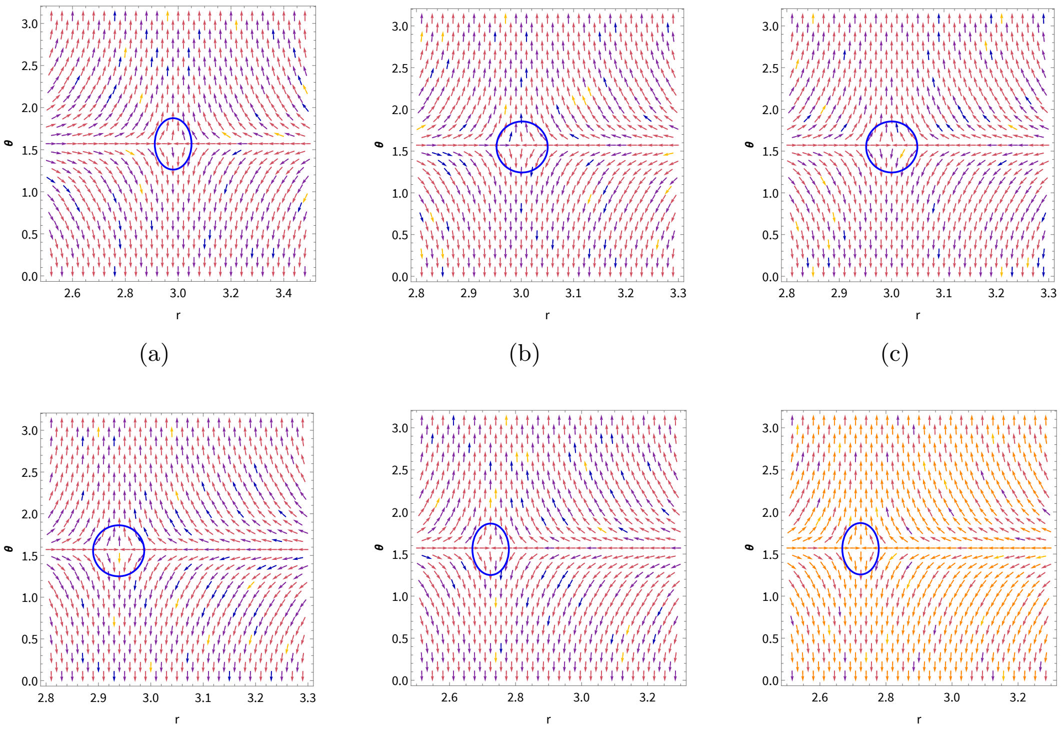
By combining Eqs. (4.6) and (4.8), one obtains explicit forms for the vector field components governing the photon sphere:

$$\phi^r = -\frac{\csc(\theta)\left(2Q_m^2 r^3\left(3M\xi + r^3(3-16\pi\xi P) + 6\xi r\right) + 3r^7(r-3M) + 12\xi^2 Q_m^4\right)}{3\left(2\xi Q_m^2 r + r^5\right)^2}, \tag{4.12}$$

and

$$\phi^\theta = -\frac{\cot(\theta)\csc(\theta)\sqrt{\frac{r^4\left(-\frac{2M}{r}+\frac{8}{3}\pi P r^2+\frac{Q_m^2}{r^2}\right)}{2\xi Q_m^2+r^4}+1}}{r^2}. \tag{4.13}$$

Within the context of thermodynamic topology, we explore the properties of photon spheres (PSs) and their significance in characterizing black hole stability. According to a fundamental topological principle, any zero of the vector field enclosed by a closed contour contributes a topological charge equal to its winding number. Each photon sphere is therefore associated with a charge of either (+1) or (-1), determined by its orientation and the direction of winding. When summing the contributions of all zero points within a given region, the total topological charge may assume values of (-1), (0), or (+1). To understand how this structure adapts to changes in system parameters, we study the photon sphere configurations across various values of ($\xi = 0.1, 1, 4$) and ($Q_m = 0.1, 0.3, 0.6$). As illustrated in Fig. 2, the total topological charge of photon spheres remains (-1) for all these parameter combinations. This indicates that the topological characteristics of the system are largely insensitive to modifications in the symmetry-breaking scale or coupling strength. Examining photon sphere arrangements in detail allows us to gain insight into the interplay between geometry and thermodynamic behavior of black holes. By analyzing the effective photon trajectories, we can assess how variations in (Q_m) and (ξ) influence the topological properties of the photon spheres. Furthermore, unstable photon spheres display unique signatures that reflect deeper links between black hole thermodynamics and spacetime curvature.



(a) (b) (c) (d) (e) (f)

Figure 2: The unit vector field in the (r, θ) plane, corresponding to the photon sphere configurations, is analyzed for the parameter combinations ($\xi = 0.1, 1, 4$) and ($Q_m = 0.1, 0.3, 0.6$)

5 Conclusion

In this work, we have conducted a comprehensive analysis of the thermodynamic topology of magnetic AdS black holes, focusing on the configuration and behavior of topological charges. Our investigation demonstrates that the vector field associated with the system's free energy provides a clear depiction of the topological structure, with zero points corresponding to distinct topological charges. We observed that for smaller values of the parameters, multiple topological charges emerge, whereas higher values lead to a single dominant charge. Across all examined parameter regimes, the total topological charge consistently aligns with the characteristics of AdS Reissner-Nordström black holes, confirming the robustness of this classification. The stability of these configurations is further supported by winding number analysis around the zero points, providing a reliable method for identifying stable and unstable phases.

In parallel, we explored the properties of photon spheres and their role in determining black hole stability. Each photon sphere carries a topological charge determined by the behavior of the vector field around its zero points, and the sum of these charges remains invariant across variations in key system parameters. Detailed examination of photon sphere arrangements also revealed unique signatures associated with unstable configurations, highlighting the intricate interplay between spacetime geometry and thermodynamic behavior.

Overall, our findings establish a systematic framework for understanding black hole stability and phase transitions through the lens of thermodynamic topology. By linking topological charges with photon sphere dynamics, this study provides deeper insights into the geometric and thermodynamic properties of black holes, offering potential implications for gravitational concepts and astrophysical observations. These results underscore the utility of topological analysis as a powerful tool for characterizing black hole configurations across diverse parameter spaces.

Authors' Contributions

All authors have the same contribution.

Data Availability

Data sharing is not applicable to this article, as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] J. D. Bekenstein, "Black holes and the second law", Jacob Bekenstein: the conservative revolutionary, 303 (2020), DOI: https://doi.org/10.1142/9789811203961_0022

[2] D. Kubizňák, R. B. Mann, M. Teo, "Black hole chemistry: thermodynamics with Lambda", Classical and Quantum Gravity **34**(6), 063001 (2017), DOI: 10.1088/1361-6382/aa5c69

[3] S. W. Hawking, D. N. Page, "Thermodynamics of black holes in anti-de Sitter space", Communications in Mathematical Physics **87**, 577 (1983), DOI: https://doi.org/10.1007/BF01208266

[4] J. Sadeghi et al., "Exploring the phase transition in charged Gauss–Bonnet black holes: a holographic thermodynamics perspectives", General Relativity and Gravitation **56**(8), 93 (2024), DOI: https://doi.org/10.1007/s10714-024-03285-x

[5] S.-W. Wei, Y.-X. Liu, "Topology of black hole thermodynamics", Phys. Rev. D **105**(10), 104003 (2022), DOI: https://doi.org/10.1103/PhysRevD.105.104003

[6] S.-W. Wei, Y.-X. Liu, R. B. Mann, "Black hole solutions as topological thermodynamic defects", Phys. Rev. Lett. **129**(19), 191101 (2022), DOI: https://doi.org/10.1103/PhysRevLett.129.191101

[7] J. Sadeghi et al., "Bardeen black hole thermodynamics from topological perspective", Annals of Physics **455**, 169391 (2023), DOI: https://doi.org/10.1016/j.aop.2023.169391

[8] J. Sadeghi et al., "Bulk-boundary and RPS Thermodynamics from Topology perspective", Chinese Physics C (2024), DOI: 10.1088/1674-1137/ad53b9

[9] Y. Sekhmani et al., "Thermodynamic topology of Black Holes in $F(R)$-Euler-Heisenberg gravity's Rainbow", (2024), DOI: https://doi.org/10.48550/arXiv.2409.04997

[10] N. J. Gogoi, P. Phukon, "Thermodynamic topology of 4D dyonic AdS black holes in different ensembles", Phys. Rev. D **108**(6), 066016 (2023), DOI: https://doi.org/10.1103/PhysRevD.108.066016

[11] B. Hazarika, P. Phukon, "Thermodynamic topology of $D = 4, 5$ Horava Lifshitz black hole in two ensembles", Nuclear Physics B **1006**, 116649 (2024), DOI: https://doi.org/10.1016/j.nuclphysb.2024.116649

[12] J. Sadeghi et al., "Topology of Hayward-AdS black hole thermodynamics", Physica Scripta **99**(2), 025003 (2024), DOI: 10.1088/1402-4896/ad186b

[13] J. Sadeghi et al., "Thermodynamic topology and photon spheres in the hyperscaling violating black holes", Astroparticle Physics **156**, 102920 (2024), DOI: https://doi.org/10.1016/j.astropartphys.2023.102920

[14] B. Hazarika, P. Phukon, "Thermodynamic topology of black holes in $f(R)$ gravity", Progress of Theoretical and Experimental Physics **2024**(4), 043E01 (2024), DOI: https://doi.org/10.1093/ptep/ptae035

[15] B. Hazarika, B. Eslam Panah, P. Phukon, "Thermodynamic topology of topological charged dilatonic black holes", (2024), DOI: https://doi.org/10.1140/epjc/s10052-024-13598-5

[16] H. Chen et al., "Thermal, topological, and scattering effects of an AdS charged black hole with an antisymmetric tensor background", (2024), DOI: https://doi.org/10.48550/arXiv.2408.03090

[17] J. Sadeghi, M. A. S. Afshar, "The role of topological photon spheres in constraining the parameters of black holes", Astroparticle Physics 102994 (2024), DOI: https://doi.org/10.1016/j.astropartphys.2024.102994

[18] S.-H. Dong et al., "Thermodynamic properties and topological charge of a static black hole in loop quantum gravity", Physics of the Dark Universe 101962 (2025), DOI: https://doi.org/10.1016/j.dark.2025.101962

[19] M. A. S. Afshar, J. Sadeghi, "Mutual Influence of Photon Sphere and Non-Commutative Parameter in Various Non-Commutative Black Holes: Part I-Towards evidence for WGC", (2024), DOI: https://doi.org/10.1016/j.dark.2025.101814

[20] B. Eslam Panah, B. Hazarika, P. Phukon, "Thermodynamic topology of topological black hole in $F(R)$-ModMax gravity's rainbow", Progress of Theoretical and Experimental Physics **2024**(8), 083E02 (2024), DOI: https://doi.org/10.1093/ptep/ptae116

[21] J. Sadeghi et al., "Bulk-boundary and RPS Thermodynamics from Topology perspective", Chinese Physics C (2024), DOI: 10.1088/1674-1137/ad53b9

[22] J. Sadeghi et al., "Thermodynamic Topology of Quantum Corrected AdS-Reissner-Nordstrom Black Holes in Kiselev Spacetime", Chinese Physics C (2024), DOI: 10.1088/1674-1137/ad711b

[23] B. Hazarika, P. Phukon, "Thermodynamic Properties and Shadows of Black Holes in $f(R,T)$ Gravity", (2024), DOI: https://doi.org/10.15302/frontphys.2025.035201

[24] F. Hosseinifar et al., "Quasinormal modes and topological characteristics of a Schwarzschild black hole surrounded by the Dehnen type dark matter halo", EPJ C **85**(8), 819 (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14549-4

[25] S.-H. Dong et al., "Thermodynamic properties and topological charge of a static black hole in loop quantum gravity", Physics of the Dark Universe 2025, 101962, DOI: https://doi.org/10.1016/j.dark.2025.101962

[26] A. B. Brzo et al., "Thermodynamic Topology of AdS Black Holes within Non-Commutative Geometry and Barrow Entropy", Nuclear Physics B 116840 (2025), DOI: https://doi.org/10.1016/j.nuclphysb.2025.116840

[27] S. N. Gashti et al., "Thermodynamic topology and photon spheres of dirty black holes within non-extensive entropy", Physics of the Dark Universe 101833 (2025), DOI: https://doi.org/10.1016/j.dark.2025.101833

[28] M. A. S. Afshar et al., "Topological Insights into Black Hole Thermodynamics: Non-Extensive Entropy in CFT framework", EPJ C **85**(4), 457 (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14173-2

[29] S. N. Gashti, B. Pourhassan, İ. Sakallı, "Thermodynamic topology and phase space analysis of AdS black holes through non-extensive entropy perspectives", EPJ C **85**(3), 305 (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14035-x

[30] S. N. Gashti, "Topology of Holographic Thermodynamics within Non-extensive Entropy", JHAP **4**(4), 59 (2024), DOI: https://doi.org/10.22128/jhap.2024.907.1101

[31] M. R. Alipour et al., “Topological classification and black hole thermodynamics”, Physics of the Dark Universe **42**, 101361 (2023), DOI: https://doi.org/10.1016/j.dark.2023.101361

[32] A. Anand, S. N. Gashti, “Universality relation and thermodynamic topology with three-parameter entropy model”, Physics of the Dark Universe (2025), DOI: https://doi.org/10.1016/j.dark.2025.101916

[33] S. N. Gashti, B. Pourhassan, “Non-extensive Entropy and Holographic Thermodynamics: Topological Insights”, EPJ C **85**, 435 (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14152-7

[34] M. A. S. Afshar, J. Sadeghi, “WGC as WCCC protector: The synergistic effects of various parameters in non-commutative black holes for identifying WGC candidate models”, Nuclear Physics B **1014**, 116872 (2025), DOI: https://doi.org/10.1016/j.nuclphysb.2025.116872

[35] M. A. S. Afshar, J. Sadeghi, “Mechanisms behind the Aschenbach effect in non-rotating black hole spacetime”, Annals of Physics (2025), DOI: https://doi.org/10.1016/j.aop.2025.169953

[36] N. Heidari, I. P. Lobo, V. B. Bezerra, “Gravitational signatures of a nonlinear electrodynamics in $f(R,T)$ gravity”, (2025), DOI: https://doi.org/10.1088/1475-7516/2025/09/015

[37] S.-W. Wei, “Topological charge and black hole photon spheres”, Phys. Rev. D **102**(6), 064039 (2020), DOI: https://doi.org/10.1103/PhysRevD.102.064039

[38] P. V. P. Cunha, E. Berti, C. A. R. Herdeiro, “Light-ring stability for ultracompact objects”, Physical Review Letters **119**(25), 251102 (2017), DOI: https://doi.org/10.1103/PhysRevLett.119.251102

[39] S. Noori Gashti et al., “Assessing WGC Compatibility in ModMax Black Holes via Photon Spheres Analysis and WCCC Validation”, (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14890-8

[40] M. R. Alipour et al., “Weak gravity conjecture validation with photon spheres of quantum corrected Reissner–Nordstrom–AdS black holes in Kiselev spacetime”, EPJ C **85**(2), 138 (2025), DOI: https://doi.org/10.1140/epjc/s10052-025-14890-8

[41] J. Sadeghi, M. A. S. Afshar, “The role of topological photon spheres in constraining the parameters of black holes”, Astroparticle Physics **162**, 102994 (2024), DOI: https://doi.org/10.1016/j.astropartphys.2024.102994

[42] M. R. Alipour et al., “Reconciling the Weak Gravity and Weak Cosmic Censorship Conjectures in Einstein-Euler-Heisenberg-AdS Black Holes”, (2025), DOI: https://doi.org/10.48550/arXiv.2504.03453

[43] A. B. Balakin, J. P. S. Lemos, A. E. Zayats, “Regular nonminimal magnetic black holes in spacetimes with a cosmological constant”, Phys. Rev. D **93**(2), 024008 (2016), DOI: https://doi.org/10.1103/PhysRevD.93.024008

[44] R. H. Ali, Z.-Y. Tang, X.-M. Kuang, “Probing thermodynamic phase transitions by dynamics of timelike particle around a magnetic AdS black hole”, (2025), DOI: https://doi.org/10.48550/arXiv.2510.13552