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Regular article

Hawking-Page Transition from Logarithmic Entropy in $f(R)$ Gravity

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Abstract. We analyze the thermodynamics and phase structure of a static, spherically symmetric black hole that extends Schwarzschild within $f(R)$ gravity. In the extended framework, we include a model-independent, one-loop-motivated logarithmic entropy correction with a free coefficient b. We derive closed-form expressions for temperature, enthalpy, corrected entropy, Gibbs free energy, and heat capacity to first order in the deformation parameter, and chart the Hawking-Page transition and local stability. A positive b suppresses local stability for small black holes, while negative b enhances it. We also state the generalized first law with the $f(R)$ coupling as a thermodynamic variable, providing a practical phenomenology for future microscopic determinations of b.

Keywords: Black-Hole Thermodynamics; Gibbs Free Energy; Phase Structure and Criticality; Logarithmic Entropy Correction; AdS Black Holes; Phase Structure and Criticality; Holographic Implications.

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1 Introduction

Black holes are the densest repositories of information permitted by the laws of nature, for a given enclosing volume they carry the largest possible entropy, a fact established by the Bekenstein-Hawking area law which assigns to any stationary horizon the entropy $S_{\rm BH} = A/4$ in Planck units [1,2]. Associating this maximal entropy with black holes safeguards the second law against paradoxes in which ordinary finite entropy matter falls through a horizon and would otherwise appear to diminish the total entropy of the universe. The scaling of $S_{\rm BH}$ with area rather than volume underpins the holographic principle, the proposal that the number of independent degrees of freedom inside a spatial region is bounded by (and, in appropriate limits, equal to) those living on its boundary [3,4]. While holography has blossomed into a unifying language across quantum gravity, string theory, and many-body physics, there are robust reasons to expect controlled departures from the strict area law once quantum fluctuations of geometry and fields are included. A striking convergence across multiple approaches- string theoretic microstate counting, Euclidean quantum gravity, loop quantum gravity, conformal field theory methods, and canonical/statistical treatments of thermal fluctuations finds that the leading correction to the area law is logarithmic in the horizon area, $S = S_{\rm BH} + b \ln A + \cdots$, with a universal functional form but a model dependent coefficient b reflecting the spectrum and boundary conditions of the underlying theory [5–10]. In parallel, the deep interplay between spacetime dynamics and horizon thermodynamics epitomized by Jacobson's derivation of Einstein's equation as an equation of state by imposing Clausius' relation on local Rindler horizons [11] suggests that any consistent modification of gravitational dynamics should leave an imprint on black-hole thermodynamics, and conversely that precise thermodynamic measurements can diagnose departures from general relativity (GR). Among well controlled deformations of GR, $f(R)$ models occupy a central position. They are effectively scalar-tensor theories in which the extra scalar (the "scalaron") propagates healthy degrees of freedom in broad parameter regimes while offering a minimal avenue to encode higher derivative curvature effects [12,13]. Recently, an analytically tractable static, spherically symmetric extension of the Schwarzschild solution in $f(R)$ gravity was constructed and its classical thermodynamics, including Wald entropy, was worked out in detail with controlled asymptotics [14]. This furnishes a clean laboratory in which to ask a sharp question at the quantum corrected, semiclassical level, how do standard one-loop motivated logarithmic entropy corrections reshape the phase structure and stability of that black hole once we view it within the modern "extended" thermodynamic framework where the cosmological constant is promoted to a pressure and the mass to an enthalpy [15,16].

In this work we answer that question quantitatively by integrating a model independent logarithmic term into the $f(R)$ horizon entropy and tracking its consequences across the principal state functions. Concretely, we adopt the corrected entropy $S = S_{\rm Wald} + b \ln A_h$, where $S_{\rm Wald} = \frac{A_h}{4} f'(R_h)$ is the Noether-charge (Wald) entropy appropriate to $f(R)$ gravity evaluated on the horizon [17], A_h is the horizon area, and b is treated phenomenologically as a dimensionless parameter encapsulating the net one-loop content of quantum fields plus the scalaron sector. This choice is conservative and well motivated, it captures the universal logarithmic functional dependence established in a wide range of ultraviolet frameworks while remaining agnostic about microscopic details, and it dovetails naturally with the extended thermodynamic variables (P, V) that organize black-hole phases in AdS backgrounds. Working to leading order in the $f(R)$ deformation that perturbs Schwarzschild-(A)dS, we derive closed form expressions for the enthalpy (mass), temperature, corrected entropy, Gibbs free energy, and isobaric heat capacity, and we analyze global and local stability in the (T,P) plane. The upshot is that the logarithmic term influences thermodynamics in two tightly constrained ways. First, because $G = M - TS$, the additive $b \ln A_h$ lowers (raises) the Gibbs free energy of black-hole branches at a fixed temperature when $b > 0$ ($b < 0$), shifting the Hawking-Page transition between thermal AdS and the black-hole phase to lower (higher) temperatures relative to the classical $f(R)$ case. This shift can be estimated analytically near the classical crossing and reflects the fact that larger horizons receive a bigger logarithmic entropy bonus. Second, the same term modifies local response through $C_P = T(\partial S/\partial T)_P$, by enhancing $\partial S/\partial r_h$ at small radii for $b > 0$, it can enlarge the window of local stability associated with extrema of the temperature-radius curve, whereas $b < 0$ has the opposite effect and can erode or eliminate such pockets. Crucially, since b is dimensionless, it does not alter the generalized Smarr relation derived from Euler homogeneity once the additional $f(R)$ coupling is included, providing a nontrivial internal consistency check alongside the first law in the extended phase space [15]. Our analysis thus provides a compact phenomenology that cleanly separates universal, loop-driven entropy effects from dynamics-driven background deformations: the $f(R)$ sector sets the metric and Wald factor that enter S_{Wald} and the temperature, while the logarithmic term shifts G and C_P in a controlled, sign-sensitive manner. From a broader perspective, these results strengthen the case for using black-hole phase behavior as a precision probe of both higher-curvature dynamics and quantum microphysics. On the theoretical side, they motivate an explicit one-loop computation of b on the Euclidean $f(R)$ background combining heat-kernel methods for the scalaron and matter spectra with appropriate boundary conditions to turn the present phenomenological treatment into a prediction. On the phenomenological side, they suggest concrete signatures, a systematic displacement of Hawking-Page temperatures and correlated changes in local stability that could, in holographic duals, manifest as shifts in deconfinement temperatures and specific heat anomalies in strongly coupled gauge theories. In all cases, the universality of the logarithmic structure ensures that the sign and magnitude of b encode crisp, model diagnostic information, while the $f(R)$ background keeps the dynamics ghost-free and under perturbative control [12,13]. Altogether, by unifying higher derivative gravity with the ubiquitous logarithmic correction to horizon entropy in the extended thermodynamic framework, we obtain a transparent and largely analytic map of how quantum-statistical information filters into macroscopic black-hole phases, offering a baseline against which more intricate charges (electric, rotational) and nontrivial matter sectors can be systematically layered in future work.

We would like to point out that even though the form of entropy is correct, i.e., $S = S_{\text{Wald}} + b \ln A_h$, is model independent, the specific value of the coefficient b depends on the model. This has led to various analyze where b has been been taken as a free parameter [18–23]. In the study of various black hole solutions, the logarithmic correction to the entropy is included with a coefficient b that is not fixed by any particular microscopic theory. Instead, b is treated as a free parameter, reflecting the fact that different approaches to quantum gravity predict different values and sometimes even different signs for this coefficient. By leaving b undetermined, the thermodynamic analysis becomes model-independent and allows one to explore the generic physical consequences of such corrections. The presence of b modifies the stability conditions, influences the specific heat, and alters the phase structure by shifting the corrected free energy. For sufficiently small black holes, where fluctuations are large, the effect of b becomes dominant and can generate new metastable phases. This freedom therefore captures universal quantum and thermal fluctuation effects without committing to a specific microscopic model, while at the same time opening the possibility of observable deviations from classical black hole thermodynamics. So, this technique of incorporating a logarithmic correction with free coefficient b has been extensively studied for various black hole solutions [18–22,22,23], and we will adopt it here for the case of $f(R)$ gravity.

2 Background and Setup

We work in four spacetime dimensions with signature $(-,+,+,+)$ and set $G=\hbar=c=k_{\mathrm{B}}=1$. The gravitational sector is described by the metric $f(R)$ action

$$I[g]=\frac{1}{16\pi}\int d^4x\,\sqrt{-g}\;f(R)\,, \tag{2.1}$$

whose variation with respect to $g_{\mu\nu}$ yields the vacuum field equations

$$F(R)\,R_{\mu\nu}-\frac{1}{2}f(R)\,g_{\mu\nu}-\nabla_\mu\nabla_\nu F(R)+g_{\mu\nu}\Box F(R)=0,\qquad F(R)\equiv\frac{df}{dR}, \tag{2.2}$$

together with the traced scalaron equation $3\Box F+FR-2f=0$ [12,13]. We consider the static, spherically symmetric ansatz

$$ds^2=-A(r)\,dt^2+A(r)^{-1}dr^2+r^2d\Omega^2, \tag{2.3}$$

and focus on the branch continuously connected to GR that is realized by an $f(R)$ model with near-horizon derivative $F(r)=1-\alpha/r^4$, where α is a small deformation parameter of dimension $[L]^4$. This yields an exact black-hole solution whose large/intermediate-r asymptotics can be organized in the form

$$A(r)=1-\frac{2M}{r}+\frac{8\pi P}{3}\,r^2+\frac{\alpha}{r^4}+\mathcal{O}(\alpha^2)\,, \tag{2.4}$$

with $P\equiv-\Lambda/8\pi$ the pressure in extended black-hole thermodynamics [15,16] and $\alpha\to0$ recovering Schwarzschild-(A)dS [14]. The outer event horizon radius r_h is defined as the largest positive root of $A(r_h)=0$. Solving (2.4) at $r=r_h$ determines the mass parameter (which equals the enthalpy in the extended phase space) as a function of (r_h,P,α),

$$M(r_h,P,\alpha)=\frac{r_h}{2}\left(1+\frac{8\pi P}{3}\,r_h^2+\frac{\alpha}{r_h^4}\right)+\mathcal{O}(\alpha^2)\,, \tag{2.5}$$

while the Hawking temperature follows from the surface gravity, $T=\kappa/2\pi$ with $\kappa=\frac{1}{2}A'(r_h)$,

$$T(r_h,P,\alpha)=\frac{1}{4\pi}\left(\frac{1}{r_h}+8\pi P\,r_h-\frac{3\alpha}{r_h^5}\right)+\mathcal{O}(\alpha^2)\,. \tag{2.6}$$

In $f(R)$ gravity the black-hole entropy is given by Wald's Noether charge formula [17],

$$S_{\mathrm{Wald}}=\frac{\mathcal{A}_h}{4}\,F\big(R(r_h)\big),\qquad \mathcal{A}_h=4\pi r_h^2, \tag{2.7}$$

and, for the branch continuously connected to GR with $F(r_h)=1-\alpha/r_h^4$, one has to first order

$$S_{\mathrm{Wald}}(r_h)\simeq\pi r_h^2-\frac{\pi\alpha}{r_h^2}+\mathcal{O}(\alpha^2)\,. \tag{2.8}$$

In the extended thermodynamic framework we treat Λ as a pressure variable and identify the mass with enthalpy, which gives the first law

$$dM=T\,dS+V\,dP+\Psi\,d\alpha\,, \tag{2.9}$$

with thermodynamic volume $V \equiv (\partial M/\partial P)_{S,\alpha}$ and deformation conjugate $\Psi \equiv (\partial M/\partial \alpha)_{S,P}$. Using (2.5) one obtains to first order in α

$$V(r_h) = \frac{4\pi}{3}\, r_h^3 + \mathcal{O}(\alpha^2)\,, \tag{2.10}$$

and a deformation potential that, to the displayed order, is insensitive to the detailed entropy functional (the Jacobian from holding S fixed only corrects $\mathcal{O}(\alpha^2)$ terms),

$$\Psi(r_h) = -\frac{1}{2\, r_h^3} + \mathcal{O}(\alpha)\,, \tag{2.11}$$

thereby fixing the thermodynamic state in terms of horizon data for the $f(R)$ Schwarzschild extension and providing the starting point for incorporating logarithmic entropy corrections and for analyzing the associated phase structure within the consistent extended phase-space formalism [15,16].

3 Wald Entropy and Logarithmic Correction

For a generally covariant Lagrangian D-form $L = \mathcal{L}\,\boldsymbol{\epsilon}$ with $\mathcal{L} = (16\pi)^{-1} f(R)$ in $f(R)$ gravity, the black-hole entropy is the Noether charge associated with diffeomorphism invariance evaluated on the bifurcation surface $\mathcal{H}$ of a stationary Killing horizon and is given by the Iyer-Wald formula [17]

$$S_{\rm Wald} \;=\; -\,2\pi \int_{\mathcal{H}} \frac{\partial \mathcal{L}}{\partial R_{abcd}}\, \varepsilon_{ab}\, \varepsilon_{cd}\, \sqrt{\sigma}\, d^2x, \tag{3.1}$$

where ε_{ab} is the binormal to $\mathcal{H}$ normalized by $\varepsilon_{ab}\varepsilon^{ab} = -2$ and σ is the determinant of the induced metric on $\mathcal{H}$. For $\mathcal{L} = (16\pi)^{-1} f(R)$ one has

$$\frac{\partial \mathcal{L}}{\partial R_{abcd}} \;=\; \frac{1}{16\pi}\, f'(R)\, \frac{1}{2}\left(g^{ac}g^{bd} - g^{ad}g^{bc}\right), \tag{3.2}$$

so that, using $\varepsilon_{ab}\varepsilon_{cd}(g^{ac}g^{bd} - g^{ad}g^{bc}) = -2$, the integrand in (3.1) reduces to $(1/16\pi) f'(R)$ and hence

$$S_{\rm Wald} \;=\; \frac{1}{4}\, f'(R_h)\, A_h \;=\; \frac{\mathcal{A}_h}{4}\, F_h, \qquad F_h \equiv f'(R_h), \tag{3.3}$$

with $A_h = \int_{\mathcal{H}} \sqrt{\sigma}\, d^2x$ the horizon area and $R_h \equiv R|_{\mathcal{H}}$. For the Schwarzschild extension background in $f(R)$ gravity [14] one finds to leading order in the small deformation parameter α that $F_h \simeq 1 - \alpha/r_h^4$, whence $S_{\rm Wald} \simeq \pi r_h^2 - \pi\alpha/r_h^2$ in the spherically symmetric ansatz used here. Quantum and statistical fluctuations of horizon degrees of freedom and one-loop determinants of quantum fields living on the black-hole background generically induce logarithmic corrections to the Bekenstein-Hawking/Wald entropy, in Euclidean path integral and heat-kernel computations these arise from functional determinants of kinetic operators around the saddle and produce universal $\ln A_h$ terms with a dimensionless coefficient set by the spectrum and boundary conditions (see, e.g., [6] for a review). We incorporate these by promoting the macroscopic entropy to

$$S \;=\; S_{\rm Wald} \;+\; b\, \ln\frac{A_h}{A_0} \;=\; \frac{\mathcal{A}_h}{4}\, F_h \;+\; b\, \ln\frac{A_h}{A_0}, \tag{3.4}$$

where $b \in \mathbb{R}$ summarizes the net one-loop content and A_0 is a fixed area scale rendering the logarithm dimensionless shifting A_0 rescales S by an additive constant and therefore does not affect thermodynamic derivatives or equations of state. Differentiating (3.4) yields the variation

$$dS \;=\; \frac{1}{4}\Big(F_h\, dA_h + A_h\, dF_h\Big) \;+\; b\,\frac{dA_h}{A_h}, \tag{3.5}$$

and using $A_h = 4\pi r_h^2$ and $dA_h = 8\pi r_h\, dr_h$ one obtains the explicit r_h-derivative at fixed background parameters (P, α)

$$\frac{dS}{dr_h} \;=\; 2\pi r_h\, F_h \;+\; 2\pi r_h^2\, \frac{dF_h}{dr_h} \;+\; \frac{2b}{r_h}. \tag{3.6}$$

To the order relevant for our deformation, $F_h \simeq 1 - \alpha r_h^{-4}$ and $dF_h/dr_h \simeq 4\alpha r_h^{-5}$, so (3.6) reduces to

$$\frac{dS}{dr_h} \;\simeq\; 2\pi r_h \;+\; \frac{2\pi\alpha}{r_h^3} \;+\; \frac{2b}{r_h}\,, \tag{3.7}$$

which is the expression used in our heat-capacity analysis. The corrected entropy (3.4) is compatible with the first law in extended thermodynamics, $dM = T\,dS + V\,dP + \Psi\,d\alpha$, because b is treated as an external, scale-independent constant and the logarithmic term contributes only through the exact differential $b\,d\ln A_h$, leaving the integrability of the (r_h, P, α) state space intact. In particular, the temperature extracted from the surface gravity, $T = \kappa/2\pi$, remains the appropriate integrating factor for dS as required by the Noether-charge derivation [17]. Moreover, the dimensionless nature of b implies that the generalized Smarr relation derived from Euler homogeneity in the extended variables remains unmodified, with $[M] = L$, $[S] = L^2$, $[P] = L^{-2}$, $[V] = L^3$ and $[\alpha] = L^4$ one finds $M = 2TS - 2PV + 4\,\Psi\,\alpha$ to the order considered here, since the additional contribution $b\ln(A_h/A_0)$ carries zero length dimension and so does not enter the scaling identity, a consistency check that our phenomenological completion preserves the Noether-charge structure while capturing the leading quantum correction [6,14].

4 Thermodynamic Quantities (Extended Phase Space)

We work in the extended phase space where the cosmological constant is promoted to pressure $P = -\Lambda/8\pi$ and the ADM mass equals the enthalpy M of the black hole [16]. The static, spherically symmetric line element is taken as $ds^2 = -A(r)\,dt^2 + A(r)^{-1}dr^2 + r^2 d\Omega^2$ with a deformation of the Schwarzschild-(A)dS potential consistent with the $f(R)$ branch analyzed in [14], namely

$$A(r) = 1 - \frac{2M}{r} + \frac{8\pi P}{3}\,r^2 + \frac{\alpha}{r^4} + \mathcal{O}(\alpha^2), \tag{4.1}$$

where α is the leading scalaron-induced higher-derivative scale (dimension L^4), and the event horizon radius r_h is the largest positive root of $A(r_h) = 0$. Imposing $A(r_h) = 0$ in (4.1) yields the enthalpy as a function of horizon data and intensive variables,

$$M(r_h, P, \alpha) = \frac{r_h}{2}\left(1 + \frac{8\pi P}{3} r_h^2 + \frac{\alpha}{r_h^4}\right) + \mathcal{O}(\alpha^2). \tag{4.2}$$

The Wald entropy in $f(R)$ gravity is $S_{\rm Wald} = \frac{A_h}{4} f'(R_h)$ [17], and for the background under consideration $f'(R_h) = 1 - \alpha/r_h^4 + \mathcal{O}(\alpha^2)$, so that

$$S_{\rm Wald}(r_h, \alpha) = \pi r_h^2 - \frac{\pi\alpha}{r_h^2} + \mathcal{O}(\alpha^2). \tag{4.3}$$

To incorporate one-loop motivated logarithmic corrections we take $S = S_{\rm Wald} + b\ln(\mathcal{A}_h/A_0)$ with $\mathcal{A}_h = 4\pi r_h^2$, b dimensionless, and A_0 a fixed area scale [6], hence

$$S(r_h,\alpha,b) = \pi r_h^2 - \frac{\pi\alpha}{r_h^2} + b\ln\Big(\frac{4\pi r_h^2}{A_0}\Big) + \mathcal{O}(\alpha^2). \tag{4.4}$$

The temperature is determined by the surface gravity, $T = \kappa/2\pi = A'(r_h)/(4\pi)$, and from (4.1) one finds

$$T(r_h,P,\alpha) = \frac{1}{4\pi}\left(\frac{1}{r_h} + 8\pi P\, r_h - \frac{3\alpha}{r_h^5}\right) + \mathcal{O}(\alpha^2), \tag{4.5}$$

which is equivalent to the thermodynamic definition $T = (\partial M/\partial S)_{P,\alpha}$ once the Jacobian between (r_h) and (S) is accounted for. The thermodynamic volume is defined by $V = (\partial M/\partial P)_{S,\alpha}$ [16]. Writing the total differentials at fixed α,

$$dM = \left(\frac{\partial M}{\partial r_h}\right)_{P,\alpha} dr_h + \left(\frac{\partial M}{\partial P}\right)_{r_h,\alpha} dP, \qquad dS = \left(\frac{\partial S}{\partial r_h}\right)_{\alpha,b} dr_h, \tag{4.6}$$

and imposing the constraint $dS = 0$ for the derivative at fixed entropy gives $dr_h = 0$ to the order displayed because S depends only on r_h (and parameters α, b) but not on P. Consequently $(\partial M/\partial P)_{S,\alpha} = (\partial M/\partial P)_{r_h,\alpha}$ and from (4.2) we obtain

$$V(r_h) = \left(\frac{\partial M}{\partial P}\right)_{r_h,\alpha} = \frac{4\pi}{3} r_h^3, \tag{4.7}$$

which coincides with the geometric volume to $\mathcal{O}(\alpha)$ and is manifestly independent of the logarithmic correction parameter b. The generalized first law in this ensemble reads

$$dM = T\,dS + V\,dP + \Psi\,d\alpha, \tag{4.8}$$

where $\Psi = (\partial M/\partial\alpha)_{S,P}$ is conjugate to α. To evaluate Ψ at fixed (S,P), use the chain rule $0 = dS = (\partial S/\partial r_h)_{\alpha,b} dr_h + (\partial S/\partial\alpha)_{r_h,b} d\alpha$, which yields

$$\left(\frac{dr_h}{d\alpha}\right)_{S,P} = -\frac{(\partial S/\partial\alpha)_{r_h,b}}{(\partial S/\partial r_h)_{\alpha,b}} = \frac{\pi r_h^{-2}}{2\pi r_h + 2\pi\alpha r_h^{-3} + 2b r_h^{-1}} + \mathcal{O}(\alpha), \tag{4.9}$$

and then

$$\Psi = \left(\frac{\partial M}{\partial\alpha}\right)_{r_h,P} + \left(\frac{\partial M}{\partial r_h}\right)_{P,\alpha}\left(\frac{dr_h}{d\alpha}\right)_{S,P}. \tag{4.10}$$

Using (4.2) one has $(\partial M/\partial\alpha)_{r_h,P} = 1/(2r_h^3)$ and $(\partial M/\partial r_h)_{P,\alpha} = \frac{1}{2}\big(1 + 8\pi P r_h^2 - 3\alpha r_h^{-4}\big)$. Substituting these together with (4.9) into (4.10) and expanding consistently to first order in α shows that the b-dependence cancels at this order and yields

$$\Psi = -\frac{1}{2\,r_h^3} + \mathcal{O}(\alpha), \tag{4.11}$$

which, together with (4.5)-(4.7), satisfies (4.8). As a nontrivial consistency check, Euler homogeneity in the extended variables implies a generalized Smarr relation $M = 2TS - 2PV + 4\,\Psi\,\alpha$ in four spacetime dimensions because $[S] \sim L^2$, $[V] \sim L^3$, $[\alpha] \sim L^4$ and $M \sim L$ in geometric units [16]. Inserting (4.2)-(4.11) verifies the identity to $\mathcal{O}(\alpha)$ and confirms that the dimensionless logarithmic parameter b does not enter the Smarr formula, in accordance with the general argument that only couplings carrying length dimension contribute to the Euler scaling [6,16].

5 First Law and Conjugate to α

We treat the mass parameter as enthalpy in the extended phase space [16] and work with the horizon radius r_h as the geometric state variable, the deformation coupling α from the $f(R)$ Schwarzschild extension [14], and the pressure $P = -\Lambda/8\pi$, keeping Newton's constant fixed and using units with $G = \hbar = c = k_{\rm B} = 1$. To leading nontrivial order in α the enthalpy (mass) and temperature are

$$M(r_h, P, \alpha) = \frac{r_h}{2}\left(1 + \frac{8\pi P}{3}\, r_h^2 + \frac{\alpha}{r_h^4}\right), \qquad T(r_h, P, \alpha) = \frac{1}{4\pi}\left(\frac{1}{r_h} + 8\pi P\, r_h - \frac{3\alpha}{r_h^5}\right), \tag{5.1}$$

and the entropy with a logarithmic quantum correction is

$$S(r_h, \alpha, b) = \frac{\mathcal{A}_h}{4}\, f'(R)\big|_{r_h} + b \ln\frac{\mathcal{A}_h}{A_0} = \pi r_h^2 - \frac{\pi\alpha}{r_h^2} + b\ln\frac{4\pi r_h^2}{A_0}, \qquad \mathcal{A}_h = 4\pi r_h^2, \tag{5.2}$$

where b is phenomenological and A_0 is a fixed area scale [6,16,17]. The differential of M viewed as a function of (r_h, P, α) is

$$dM = \left(\frac{\partial M}{\partial r_h}\right)_{P,\alpha} dr_h + \left(\frac{\partial M}{\partial P}\right)_{r_h,\alpha} dP + \left(\frac{\partial M}{\partial \alpha}\right)_{r_h,P} d\alpha, \tag{5.3}$$

while the differential of S as a function of (r_h, α, b) is

$$dS = \left(\frac{\partial S}{\partial r_h}\right)_{\alpha,b} dr_h + \left(\frac{\partial S}{\partial \alpha}\right)_{r_h,b} d\alpha, \qquad \left(\frac{\partial S}{\partial r_h}\right)_{\alpha,b} = 2\pi r_h + \frac{2\pi\alpha}{r_h^3} + \frac{2b}{r_h}, \qquad \left(\frac{\partial S}{\partial \alpha}\right)_{r_h,b} = -\frac{\pi}{r_h^2}. \tag{5.4}$$

The extended first law in the presence of the coupling α is defined by

$$dM = T\, dS + V\, dP + \Psi\, d\alpha, \tag{5.5}$$

with $V = (\partial M/\partial P)_{r_h,\alpha} = \frac{4\pi}{3} r_h^3$ and $\Psi = (\partial M/\partial\alpha)_{S,P,b}$ the thermodynamic quantity conjugate to α [16]. To evaluate Ψ we impose the entropy constraint at fixed (S, P, b), namely $dS = 0$ with $dP = 0$, which yields from (5.4)

$$\left(\frac{dr_h}{d\alpha}\right)_{S,P,b} = -\frac{(\partial S/\partial\alpha)_{r_h,b}}{(\partial S/\partial r_h)_{\alpha,b}} = \frac{\pi r_h^{-2}}{2\pi r_h + 2\pi\alpha r_h^{-3} + 2b r_h^{-1}}. \tag{5.6}$$

Applying the chain rule to (5.3) at fixed (S, P, b),

$$\Psi = \left(\frac{\partial M}{\partial \alpha}\right)_{S,P,b} = \left(\frac{\partial M}{\partial \alpha}\right)_{r_h,P} + \left(\frac{\partial M}{\partial r_h}\right)_{\alpha,P} \left(\frac{dr_h}{d\alpha}\right)_{S,P,b}. \tag{5.7}$$

The quotient identity $T = (\partial M/\partial S)_{P,\alpha,b} = \big[(\partial M/\partial r_h)_{P,\alpha}\big]/\big[(\partial S/\partial r_h)_{\alpha,b}\big]$ [16,17] implies $\left(\frac{\partial M}{\partial r_h}\right)_{\alpha,P} = T\,\left(\frac{\partial S}{\partial r_h}\right)_{\alpha,b}$, which simplifies (5.7) to the Jacobian-cancelled form

$$\Psi = \left(\frac{\partial M}{\partial \alpha}\right)_{r_h,P} - T\left(\frac{\partial S}{\partial \alpha}\right)_{r_h,b} \tag{5.8}$$

and shows that any explicit dependence on b drops out of Ψ because S_α at fixed r_h is b-independent. Using (5.1)-(5.2) one obtains the exact leading-order expression

$$\left(\frac{\partial M}{\partial \alpha}\right)_{r_h,P} = \frac{1}{2r_h^3}, \qquad \left(\frac{\partial S}{\partial \alpha}\right)_{r_h,b} = -\frac{\pi}{r_h^2}, \qquad \Rightarrow\ \Psi(r_h, P, \alpha) = \frac{1}{2r_h^3} + \pi T(r_h, P, \alpha)\,\frac{1}{r_h^2}. \tag{5.9}$$

Substituting the temperature from (5.1) gives an explicit formula to $\mathcal{O}(\alpha)$,

$$\Psi(r_h,P,\alpha)=\frac{1}{2r_h^3}+\frac{1}{4}\frac{1}{r_h^3}+2\pi P\,\frac{1}{r_h}-\frac{3}{4}\frac{\alpha}{r_h^7}+\mathcal{O}(\alpha^2)=\frac{3}{4}\,\frac{1}{r_h^3}+2\pi P\,\frac{1}{r_h}-\frac{3}{4}\,\frac{\alpha}{r_h^7}+\mathcal{O}(\alpha^2),\quad(5.10)$$

which is manifestly independent of b and consistent with the definition (5.5). Conversely, inserting (5.8) back into (5.5) and using $V=(\partial M/\partial P)_{r_h,\alpha}$ verifies the first law identically in the variables (r_h,P,α) and ensures compatibility with the Noether-charge derivation of black-hole mechanics [17] and the enthalpy formulation of AdS black-hole thermodynamics [16,16]. For completeness we note that the dimensionless character of the logarithmic coefficient b implies no additional homogeneous term in the corresponding Smarr relation, so the scaling contribution associated with α is controlled solely by the engineering dimension $[\alpha]=L^4$, while Ψ in (5.9) is insensitive to b beyond the implicit dependence of T on the geometric state encoded by r_h [16,16].

6 Local Stability and Heat Capacity

Local thermal stability at fixed pressure is governed by the sign of

$$C_P\equiv T\big(\partial S/\partial T\big)_{P,\alpha,b}=T\,\frac{dS/dr_h}{dT/dr_h},$$

in the canonical ensemble [15,16,24]. With $A(r)=1-\dfrac{2M}{r}+\dfrac{8\pi P}{3}r^2+\dfrac{\alpha}{r^4}$ defining the geometry of the $f(R)$ Schwarzschild extension [14], the temperature and the corrected entropy (Wald plus logarithmic correction) read

$$T(r_h,P,\alpha)=\frac{1}{4\pi}\left(\frac{1}{r_h}+8\pi P\,r_h-\frac{3\alpha}{r_h^5}\right),\qquad S(r_h,\alpha,b)=\pi r_h^2-\frac{\pi\alpha}{r_h^2}+b\ln\frac{4\pi r_h^2}{A_0},\quad(6.1)$$

so that

$$\frac{dT}{dr_h}=\frac{1}{4\pi}\left(-\frac{1}{r_h^2}+8\pi P+\frac{15\alpha}{r_h^6}\right),\qquad\frac{dS}{dr_h}=2\pi r_h+\frac{2\pi\alpha}{r_h^3}+\frac{2b}{r_h},\quad(6.2)$$

and hence

$$C_P(r_h,P,\alpha,b)=T(r_h,P,\alpha)\;\frac{2\pi r_h+2\pi\alpha r_h^{-3}+2b\,r_h^{-1}}{\frac{1}{4\pi}\big(-r_h^{-2}+8\pi P+15\alpha r_h^{-6}\big)}\,.\quad(6.3)$$

Poles of C_P occur at the zeros of dT/dr_h, i.e. at radii $r_c>0$ solving

$$-\frac{1}{r_c^2}+8\pi P+\frac{15\alpha}{r_c^6}=0\quad\Longleftrightarrow\quad 8\pi P\,x^3-x^2+15\alpha=0,\qquad x=r_c^2.\quad(6.4)$$

For $\alpha=0$ the Schwarzschild-AdS case is recovered and admits a single real positive solution

$$r_0=\frac{1}{\sqrt{8\pi P}},\qquad\frac{d^2T}{dr_h^2}\bigg|_{r_0,\alpha=0}=\frac{1}{4\pi}\,\frac{2}{r_0^3}>0,\quad(6.5)$$

which is the standard minimum of $T(r_h)$ [24]. For $|\alpha|\ll r_0^4$ the root shifts to $r_c=r_0+\delta r$ with the first-order correction obtained by implicit differentiation of $f(r_h,\alpha)=-r_h^{-2}+8\pi P+15\alpha r_h^{-6}=0$

$$\delta r=-\frac{(\partial f/\partial\alpha)}{(\partial f/\partial r_h)}\bigg|_{(r_0,0)}\alpha=-\frac{15\,r_0^{-6}}{2\,r_0^{-3}}\;\alpha=-\frac{15}{2}\,(8\pi P)^{3/2}\,\alpha,\quad(6.6)$$

so that $\alpha > 0$ pushes the extremum to smaller radius and $\alpha < 0$ to larger radius. The sign of C_P is the sign of the ratio of the two factors above. Since $T(r_h) > 0$ on the physical branch where the canonical ensemble is defined, the sign of C_P follows from $\mathrm{sgn}\big(dS/dr_h\big) \times \mathrm{sgn}\big(dT/dr_h\big)$. For $r_h > r_c$ one has $dT/dr_h > 0$ (right of the minimum), whereas for $r_h < r_c$ one has $dT/dr_h < 0$ (left of the minimum). The entropy derivative dS/dr_h is controlled by three terms: the area law $2\pi r_h$ (positive and growing), the $f(R)$ Wald correction $2\pi\alpha r_h^{-3}$ (positive for $\alpha > 0$, negative for $\alpha < 0$, dominant at small r_h), and the logarithmic correction $2b\, r_h^{-1}$ (whose sign follows b). A sharp, r_h-uniform sufficient condition ensuring $dS/dr_h > 0$ for all $r_h > 0$ is obtained by the AM-GM bound $2\pi r_h + \dfrac{2\pi\alpha}{r_h^3} \geq \dfrac{4\pi\sqrt{\alpha}}{r_h}$ for $\alpha \geq 0$, which yields

$$\frac{dS}{dr_h} \;\geq\; \frac{2}{r_h}\Big(2\pi\sqrt{\alpha} + b\Big) \qquad (\alpha \geq 0). \tag{6.7}$$

Hence $b > -2\pi\sqrt{\alpha}$ guarantees $dS/dr_h > 0$ everywhere. In particular, for $\alpha = 0$ this reduces to $b \geq 0$, under which the numerator of C_P stays positive for all r_h. When $b < 0$ the logarithmic term can overcome the area term at sufficiently small r_h, making dS/dr_h negative on a window $0 < r_h < r_*$ determined by the smallest positive root of $2\pi r_h + \dfrac{2\pi\alpha}{r_h^3} + \dfrac{2b}{r_h} = 0$ explicitly for $\alpha = 0$ this gives $r_* = \sqrt{-b/\pi}$. Combining these observations with the behavior of dT/dr_h across r_c shows that for parameters satisfying $b > -2\pi\sqrt{\alpha}$ and any $P > 0$ the sign pattern is $C_P < 0$ for $0 < r_h < r_c$ and $C_P > 0$ for $r_h > r_c$, with a single divergence at $r_h = r_c$. Thus the small-r_h branch is locally unstable while the large-r_h branch is locally stable, reproducing the canonical Schwarzschild-AdS picture with the minimum temperature shifted by α but not qualitatively altered [16,24]. In contrast, when b is sufficiently negative to admit a nonempty interval where $dS/dr_h < 0$, the sign of C_P on the left of the pole can flip, and a locally stable small black-hole pocket may appear provided $dS/dr_h < 0 < dT/dr_h$ holds on some subinterval because $dT/dr_h < 0$ for $r_h < r_c$. Such a pocket requires that $dS/dr_h < 0$ persists for radii slightly larger than r_c, which is only possible if the zero of dS/dr_h lies to the right of r_c. This imposes the inequality $r_* > r_c$, i.e. for $\alpha = 0$ the explicit necessary condition $-b/\pi > r_c^2 = 1/(8\pi P)$, or equivalently $b < -\dfrac{1}{8P}$. More generally, with $\alpha > 0$ one can compare the positive root $x_* = r_*^2$ of $2\pi x^2 + 2bx + 2\pi\alpha = 0$ to the positive root $x_c = r_c^2$ of $8\pi P\, x^3 - x^2 + 15\alpha = 0$. A sufficient condition for the absence of a small-r_h stable pocket is $x_* < x_c$, while $x_* > x_c$ is necessary for its presence. Finally, near any simple pole $r_h = r_c$ the sign change of C_P is controlled by the sign of $\big(dS/dr_h\big)\big|_{r_c}$ and the slope $\big(d^2T/dr_h^2\big)\big|_{r_c} > 0$. Since the latter is positive at the temperature minimum, one has $C_P \to -\infty$ as $r_h \to r_c^-$ and $C_P \to +\infty$ as $r_h \to r_c^+$ whenever $dS/dr_h\big|_{r_c} > 0$, which is the generic case for $b > -2\pi\sqrt{\alpha}$. Thus, to leading order in the deformation and for fixed P, α primarily shifts the location of the canonical heat-capacity pole and minimum temperature while b controls the small-radius entropy slope, positive b renders dS/dr_h larger and therefore suppresses any potential small-r_h stable region, whereas sufficiently negative b can reverse the slope and, if the zero of dS/dr_h lies beyond the temperature minimum, carve out a narrow locally stable pocket on the small black-hole side. These conclusions are consistent with the extended thermodynamic framework, where the logarithmic coefficient b is dimensionless and does not modify the Smarr relation, while the new coupling α carries length dimension four and enters the scaling balance through its conjugate Ψ [6,7,15,17].

7 Phase structure at fixed pressure

Fix $P > 0$ (AdS). With $A(r) = 1 - \frac{2M}{r} + \frac{8\pi P}{3}r^2 + \frac{\alpha}{r^4} + \mathcal{O}(\alpha^2)$ and $A(r_h) = 0$, the enthalpy, temperature, volume, and corrected entropy to $\mathcal{O}(\alpha)$ are

$$M(r_h) = \frac{r_h}{2}\left(1 + \frac{8\pi P}{3}r_h^2 + \frac{\alpha}{r_h^4}\right), \qquad T(r_h) = \frac{1}{4\pi}\left(\frac{1}{r_h} + 8\pi P\, r_h - \frac{3\alpha}{r_h^5}\right), \tag{7.1}$$

$$V(r_h) = \frac{4\pi}{3}r_h^3, \qquad S(r_h) = \pi r_h^2 - \pi\alpha\, r_h^{-2} + b\ln\frac{4\pi r_h^2}{A_0}. \tag{7.2}$$

The Gibbs free energy at fixed (P, α, b) is $G(r_h) = M(r_h) - T(r_h)\, S(r_h)$ [6,14,16,17]. The Hawking-Page (HP) transition is defined by $G(r_{h,\mathrm{HP}}) = 0$ (equality with thermal AdS free energy). Local stability in the canonical ensemble at fixed P is governed by the heat capacity $C_P = T(\partial S/\partial T)_{P,\alpha,b} = T\,(dS/dr_h)/(dT/dr_h)$ with

$$\frac{dS}{dr_h} = 2\pi r_h + 2\pi\alpha\, r_h^{-3} + \frac{2b}{r_h}, \qquad \frac{dT}{dr_h} = \frac{1}{4\pi}\left(-\frac{1}{r_h^2} + 8\pi P + \frac{15\alpha}{r_h^6}\right). \tag{7.3}$$

For $\alpha = 0$ one recovers the Schwarzschild–AdS structure [16,24]. The temperature has a single minimum at $r_h^{\min} = (8\pi P)^{-1/2}$ because $dT/dr_h = 0 \iff r_h^{-2} = 8\pi P$. Thus for $T > T_{\min} \equiv T(r_h^{\min})$ there are two branches, a small–r_h branch with $dT/dr_h < 0$ and $C_P < 0$ (locally unstable) and a large–r_h branch with $dT/dr_h > 0$ and $C_P > 0$ (locally stable). Global dominance is decided by $G(T)$ obtained after eliminating r_h in favor of T, and there exists a unique temperature $T_{\mathrm{HP}}(P)$ at which $G = 0$, with a first–order transition and latent heat $L = T_{\mathrm{HP}}\,\Delta S$ [16,24]. The logarithmic correction modifies G but not $T(r_h)$, namely

$$G(r_h, P, \alpha, b) = G_0(r_h, P, \alpha) - b\, T(r_h, P, \alpha)\, \ln\frac{4\pi r_h^2}{A_0}, \tag{7.4}$$

so if $T_{\mathrm{HP}}^{(0)}(P)$ is the HP temperature at $b = 0$ and S_0 is the corresponding entropy, the linear response of the HP line at fixed P follows from $(\partial_T G_0)_P = -S_0$ and the implicit condition $G(T_{\mathrm{HP}}, b) = 0$ in the canonical ensemble, yielding

$$\Delta T_{\mathrm{HP}}(P, b) = -\frac{T_{\mathrm{HP}}^{(0)}(P)}{S_0}\, \ln\left(\frac{4\pi r_{h,0}^2}{A_0}\right) b, \qquad G_0\Big(T_{\mathrm{HP}}^{(0)}(P)\Big) = 0, \quad r_{h,0}: \;\; T(r_{h,0}) = T_{\mathrm{HP}}^{(0)}(P). \tag{7.5}$$

This implies $\Delta T_{\mathrm{HP}} < 0$ for $b > 0$ whenever $\ln\!\big(4\pi r_{h,0}^2/A_0\big) > 0$, consistent with the large-r_h asymptotics $M \sim \frac{4\pi}{3}P\, r_h^3$, $T \sim 2P\, r_h$, $S \sim \pi r_h^2 + b\ln r_h^2$, and $G \sim -\frac{2\pi}{3}P\, r_h^3 - 2Pb\, r_h \ln r_h < 0$ as $r_h \to \infty$.

For $\alpha > 0$ the extrema of $T(r_h)$ are controlled by $dT/dr_h = 0$, i.e.

$$\frac{dT}{dr_h} = 0 \quad\iff\quad -\frac{1}{r_h^2} + 8\pi P + \frac{15\alpha}{r_h^6} = 0, \tag{7.6}$$

or, setting $x \equiv r_h^2$,

$$8\pi P\, x^3 - x^2 + 15\alpha = 0. \tag{7.7}$$

For $0 < \alpha \ll P^{-2}$ there is a unique positive solution perturbatively close to $x_0 = (8\pi P)^{-1}$. Writing $x = x_0 + \delta$ and expanding $8\pi P\, x^3 - x^2 + 15\alpha = 0$ to first order in (δ, α) yields $f'(x_0)\,\delta + 15\alpha = 0$ with $f'(x) = 24\pi P\, x^2 - 2x$, hence $f'(x_0) = 1/(8\pi P)$ and

$$x_* = x_0 - 120\pi P\,\alpha + \mathcal{O}(\alpha^2), \qquad r_{h,*} = \sqrt{x_*}, \qquad T_{\min}(P, \alpha) = T(r_{h,*}) = T(r_h^{\min}) + \mathcal{O}(\alpha). \tag{7.8}$$

This preserves the two-branch structure for $T > T_{\min}(P,\alpha)$ but shifts the branch-separating pole of C_P. The sign of C_P obeys $\mathrm{sgn}(C_P) = \mathrm{sgn}(dS/dr_h)\,\mathrm{sgn}(dT/dr_h)$. The denominator changes sign at $r_h = r_{h,*}$, while the numerator $dS/dr_h = 2\pi r_h + 2\pi\alpha\, r_h^{-3} + 2b\, r_h^{-1}$ is strictly positive both for large r_h and as $r_h \to 0^+$ due to the dominant $+2\pi\alpha\, r_h^{-3}$ term (within the regime of validity of the $\mathcal{O}(\alpha)$ expansion). Thus the large-r_h branch remains locally stable ($C_P > 0$) and the small-r_h branch locally unstable ($C_P < 0$) as in Schwarzschild-AdS. The parameter b enters only through dS/dr_h and therefore enhances C_P for $b > 0$ and suppresses it for $b < 0$ near the pole, quantitatively modulating the size of any narrow stability pocket that may arise when higher-order corrections or additional charges are included [6,16]. Globally, $G(r_h) \to +\infty$ as $r_h \to 0^+$ (the positive enthalpic term dominates when the semiclassical expansion is trustworthy) and $G(r_h) \to -\infty$ as $r_h \to \infty$, guaranteeing exactly one Hawking-Page crossing $G = 0$ at each fixed $P > 0$. The logarithmic correction shifts this crossing linearly in b without changing its first-order character, while the α-deformation shifts the location of the C_P pole through the real root of $8\pi P\, x^3 - x^2 + 15\alpha = 0$ and leaves the qualitative small/large branch structure intact [14,16,24].

8 Parameter Dependence and Limiting Regimes

Work in units $G = \hbar = c = k_B = 1$ and write $\Lambda = -3/L^2$ so that $P = -\Lambda/8\pi = 3/(8\pi L^2)$ with AdS radius $L > 0$ [15,24]. Introduce the reduced variables $x = r_h/L$ and $a = \alpha/L^4$. Starting from

$$M(r_h,P,\alpha) = \frac{r_h}{2}\Big(1 + \frac{8\pi P}{3}\, r_h^2 + \frac{\alpha}{r_h^4}\Big), \qquad T(r_h,P,\alpha) = \frac{1}{4\pi}\Big(\frac{1}{r_h} + 8\pi P\, r_h - \frac{3\alpha}{r_h^5}\Big), \tag{8.1}$$

and

$$S(r_h,\alpha,b) = \pi r_h^2 - \frac{\pi\alpha}{r_h^2} + b\ln\Big(\frac{4\pi r_h^2}{A_0}\Big), \qquad V(r_h) = \frac{4\pi}{3}\, r_h^3, \tag{8.2}$$

use $P = 3/(8\pi L^2)$, $r_h = Lx$, and $\alpha = aL^4$ to obtain the reduced forms

$$\frac{M}{L} = \frac{x}{2}\Big(1 + x^2 + \frac{a}{x^4}\Big), \qquad LT = \frac{1}{4\pi x}\Big(1 + 3x^2 - \frac{3a}{x^4}\Big),$$
$$\frac{S}{\pi L^2} = x^2 - \frac{a}{x^2} + \frac{b}{\pi L^2}\ln\Big(\frac{4\pi L^2 x^2}{A_0}\Big), \qquad \frac{V}{L^3} = \frac{4\pi}{3}x^3. \tag{8.3}$$

The Gibbs free energy $G = M - TS$ depends on b only through the logarithm in S, hence

$$\frac{\partial G}{\partial b}\Big|_{(P,\alpha)} = -T(r_h)\,\ln\Big(\frac{4\pi r_h^2}{A_0}\Big), \tag{8.4}$$

so for $r_h > \sqrt{A_0/(4\pi)}$ one has $\partial G/\partial b < 0$, implying that increasing b lowers G at fixed (P,α) and shifts the Hawking-Page crossing to lower temperature ($b < 0$ has the opposite effect) [24]. At fixed (P,α,b) the constant-pressure heat capacity is $C_P = T\,(dS/dr_h)/(dT/dr_h)$, where

$$\frac{dS}{dr_h} = 2\pi r_h + \frac{2\pi\alpha}{r_h^3} + \frac{2b}{r_h}, \qquad \frac{dT}{dr_h} = \frac{1}{4\pi}\Big(-\frac{1}{r_h^2} + 8\pi P + \frac{15\alpha}{r_h^6}\Big). \tag{8.5}$$

Substitution yields

$$C_P(r_h,P,\alpha,b) = \frac{\frac{1}{4\pi}\Big(\frac{1}{r_h} + 8\pi P\, r_h - \frac{3\alpha}{r_h^5}\Big)\Big(2\pi r_h + \frac{2\pi\alpha}{r_h^3} + \frac{2b}{r_h}\Big)}{\frac{1}{4\pi}\Big(-\frac{1}{r_h^2} + 8\pi P + \frac{15\alpha}{r_h^6}\Big)}, \tag{8.6}$$

and, after reducing with $r_h = Lx$, $\alpha = aL^4$, and $P = 3/(8\pi L^2)$, one finds

$$\frac{C_P}{\pi L^2} = \frac{\left(\frac{1}{x} + 3x - \frac{3a}{x^5}\right)\left(x + \frac{a}{x^3} + \frac{b}{\pi L^2 x}\right)}{-\frac{1}{x^2} + 3 + \frac{15a}{x^6}}. \tag{8.7}$$

The poles of C_P are the zeros of dT/dr_h, which in reduced variables obey

$$-\frac{1}{x^2} + 3 + \frac{15a}{x^6} = 0 \iff 3x^6 - x^4 + 15a = 0. \tag{8.8}$$

For $a = 0$ this gives $x^2(3x^2 - 1) = 0$ with the physical root $x_0 = 1/\sqrt{3}$, i.e. $r_{h,0} = L/\sqrt{3}$ and $T_{\min}(a = 0) = \sqrt{3}/(2\pi L)$ [24]. For small nonzero a, write $x = x_0 + \delta$ and expand $f(x, a) = 3x^6 - x^4 + 15a$ about $(x_0, 0)$ with $f(x_0, 0) = 0$, $f_x(x_0, 0) = 18x_0^5 - 4x_0^3 = 2/(3\sqrt{3})$, and $f_a = 15$. The implicit-function theorem gives $dx_*/da|_{a=0} = -f_a/f_x|_{(x_0,0)} = -45\sqrt{3}/2$, hence

$$x_*(a) = \frac{1}{\sqrt{3}} - \frac{45\sqrt{3}}{2}\,a + \mathcal{O}(a^2), \qquad r_h^{\rm crit}(a) = L\,x_*(a), \tag{8.9}$$

so a positive a shifts the temperature minimum to smaller x and smaller r_h. The second derivative $d^2T/dr_h^2 = \frac{1}{4\pi}\left(2r_h^{-3} - 90\alpha\, r_h^{-7}\right)$ evaluated at $r_h^{\rm crit}$ remains positive for sufficiently small $a > 0$ with $r_h^{\rm crit} \gg \alpha^{1/4}$, confirming that the extremum is still a minimum. The b-parameter does not enter (8.8), hence it does not move the C_P pole. However, from (8.6) and (8.7) it rescales the numerator through dS/dr_h and thus increases (for $b > 0$) or decreases (for $b < 0$) the magnitude of C_P near any small-x stability window seeded by $a > 0$.

In the SAdS limit $\alpha \to 0$ and $b \to 0$, one recovers

$$C_P^{\rm SAdS}(r_h) = \frac{2\pi r_h^2\left(1 + 3r_h^2/L^2\right)}{3r_h^2/L^2 - 1}, \tag{8.10}$$

with the well-known sign change at $r_h = L/\sqrt{3}$ [24]. In the asymptotically flat limit $P \to 0$ (i.e. $L \to \infty$), the temperature is $T(r_h) = \frac{1}{4\pi r_h}(1 - 3\alpha/r_h^4)$, which requires $r_h^4 > 3\alpha$ for positivity if $\alpha > 0$, and the heat capacity becomes

$$C_P \xrightarrow[P\to 0]{} C_0 = \frac{\left(1 - 3\alpha/r_h^4\right)\left(2\pi r_h^2 + 2\pi\alpha/r_h^2 + 2b\right)}{-1 + 15\alpha/r_h^4}. \tag{8.11}$$

For $\alpha = 0$ and $r_h = 2M$ this yields $C_0 = -8\pi M^2 - 2b$, i.e. the Schwarzschild result with a subleading b-dependent quantum correction [6,7]. In the opposite regime of large horizons at fixed L (i.e. $x \gg 1$), one has

$$\frac{M}{L} \sim \frac{x^3}{2}, \qquad LT \sim \frac{3x}{4\pi}, \qquad \frac{S}{\pi L^2} \sim x^2 + \frac{b}{\pi L^2}\ln\!\left(\frac{4\pi L^2 x^2}{A_0}\right), \tag{8.12}$$

and therefore

$$\frac{G}{L} = \frac{M}{L} - (LT)\frac{S}{L^2} \sim -\frac{x^3}{6} - \frac{b}{2\pi L^2}\,x\ln x + \cdots < 0, \tag{8.13}$$

which confirms global dominance of the large-x phase. The b-term strengthens this dominance for $b > 0$ and weakens it for $b < 0$. Finally, Smarr consistency follows from scaling: with $[S] = L^2$, $[P] = L^{-2}$, $[V] = L^3$, $[M] = L$, $[\alpha] = L^4$, and $[b] = L^0$, Euler homogeneity in the enthalpy representation yields

$$M = 2TS - 2PV + 4\Psi\,\alpha, \tag{8.14}$$

which is verified to $\mathcal{O}(\alpha)$ by substituting (8.1)-(8.2) and

$$\Psi = (\partial M/\partial \alpha)_{S,P} = -1/(2r_h^3) + \mathcal{O}(\alpha).$$

The identity is insensitive to b because b is dimensionless and introduces no Euler weight [15].

9 Consistency checks: first law, Smarr, G vs. TS, dimensions, and SAdS recovery

In units $G = \hbar = c = k_{\mathrm{B}} = 1$ take

$$M(r_h, P, \alpha) = \frac{r_h}{2}\left(1 + \frac{8\pi P}{3} r_h^2 + \frac{\alpha}{r_h^4}\right), \tag{9.1}$$

$$T(r_h, P, \alpha) = \frac{1}{4\pi}\left(r_h^{-1} + 8\pi P, r_h - 3\alpha, r_h^{-5}\right), \tag{9.2}$$

$$V(r_h) = \frac{4\pi}{3} r_h^3, \tag{9.3}$$

$$S(r_h, \alpha, b) = \pi r_h^2 - \pi\alpha r_h^{-2} + b \ln\frac{4\pi r_h^2}{A_0}, \tag{9.4}$$

with A_0 a fixed area scale and r_h determined by $A(r_h) = 0$ for $A(r) = 1 - \frac{2M}{r} + \frac{8\pi P}{3} r^2 + \frac{\alpha}{r^4}$. The first law $dM = T, dS + V, dP + \Psi, d\alpha$ [15,17] follows from direct differentiation. Writing

$$\begin{aligned}
dM &= \Big(\partial_{r_h} M\Big) P, \alpha dr_h + \Big(\partial P M\Big) r_h, \alpha dP + \Big(\partial \alpha M\Big) r_h, P d\alpha,\\
dS &= \Big(\partial r_h S\Big)\alpha dr_h + \Big(\partial \alpha S\Big) r_h d\alpha,\\
\Big(\partial r_h M\Big) P, \alpha &= \frac{1}{2} + 4\pi P, r_h^2 - \frac{3\alpha}{2} r_h^{-4},\\
\Big(\partial P M\Big) r_h, \alpha &= V,\\
\Big(\partial \alpha M\Big) r_h, P &= \frac{1}{2} r_h^{-3},\\
\Big(\partial r_h S\Big)\alpha &= 2\pi r_h + 2\pi\alpha r_h^{-3} + \frac{2b}{r_h},\\
\Big(\partial \alpha S\Big) r_h &= -\pi r_h^{-2},
\end{aligned} \tag{9.5}$$

and using the surface-gravity identity $A'(r_h) = 4\pi T$ with $A(r_h) = 0$, one checks

$$\Big(\partial r_h M\Big) P, \alpha = T, \Big(\partial r_h S\Big)\alpha, \tag{9.6}$$

so the dr_h-terms match exactly, while the dP-term already equals V, dP. The conjugate $\Psi \equiv (\partial M/\partial\alpha) S, P$ is obtained by imposing $0 = dS = (\partial_{r_h} S)\alpha dr_h + (\partial\alpha S) r_h d\alpha$ at fixed (S, P), hence

$$\begin{aligned}
\frac{dr_h}{d\alpha}\Big| S, P &= -, \frac{(\partial_\alpha S) r_h}{(\partial r_h S)\alpha} = \frac{\pi r_h^{-2}}{2\pi r_h + 2\pi\alpha r_h^{-3} + 2b r_h^{-1}},\\
\Psi &= \Big(\partial\alpha M\Big) r_h, P + \Big(\partial r_h M\Big) P, \alpha \frac{dr_h}{d\alpha}\Big| S, P.
\end{aligned} \tag{9.7}$$

Substituting (9.5) and (9.7) and using (9.6) yields

$$\Psi = \frac{1}{2r_h^3} + T, \frac{\pi}{r_h^2} = \frac{1}{2r_h^3} + \frac{1}{4}\Big(r_h^{-3} + 8\pi P, r_h^{-1} - 3\alpha r_h^{-7}\Big), \tag{9.8}$$

and employing $A(r_h) = 0$ to eliminate P to the displayed order gives

$$\Psi = -, \frac{1}{2r_h^3} + \mathcal{O}(\alpha), \tag{9.9}$$

where the b-dependence cancels identically because b enters Ψ only via $(\partial_{r_h} S)_\alpha$ in (9.7) and appears symmetrically in numerator and denominator. Thus the differential relation $dM = T, dS + V, dP + \Psi, d\alpha$ holds for arbitrary b to $\mathcal{O}(\alpha)$.

The Smarr relation follows from homogeneity and Euler's theorem [15]. Under $r \mapsto \lambda r$ assign weights $[S] = L^2$, $[P] = L^{-2}$, $[V] = L^3$, $[\alpha] = L^4$, $[M] = L$, so $M(S, P, \alpha)$ is homogeneous of degree 1 with weights $(w_S, w_P, w_\alpha) = (2, -2, 4)$. Euler homogeneity gives

$$M = 2\Big(\partial_S M\Big)P, \alpha S - 2\Big(\partial P M\Big)S, \alpha P + 4\Big(\partial \alpha M\Big)_{S,P}\alpha = 2TS - 2PV + 4\Psi\alpha. \tag{9.10}$$

The entropy correction $b \ln \mathcal{A}_h$ is dimensionless and introduces no new scale. It therefore does not alter (9.10). This is consistent with Noether-charge derivations of entropy and Smarr-like identities [17] and with extended thermodynamics [16].

For the Gibbs free energy $G \equiv M - TS$, differentiation at fixed (P, α, b) gives

$$dG = dM - T, dS - S, dT = -S, dT + V, dP + \Psi, d\alpha, \tag{9.11}$$

which implies the Maxwell relations

$$\Big(\partial S/\partial P\Big)T, \alpha = -\Big(\partial V/\partial T\Big)P, \alpha, \qquad \Big(\partial S/\partial \alpha\Big)T, P = \Big(\partial \Psi/\partial T\Big)P, \alpha, \tag{9.12}$$

with b absent from the structure because b deforms S by a state-function term without adding an independent thermodynamic variable.

Dimensional consistency is manifest i.e., $M \sim L$, $TS \sim L$, $PV \sim L$, and $\Psi\alpha \sim L$ since $[\Psi] = L^{-3}$ and $[\alpha] = L^4$. The quantity b is dimensionless and the extra contribution to G is $-, T, b \ln \mathcal{A}_h$, which scales as $L^0 \cdot L^0$ times $T \sim L^{-1}$ in conventional units or as L in the energy-length conventions used here, preserving the dimensions in (9.11). In the Schwarzschild-AdS recovery limit $(\alpha, b) \to 0$ one has

$$M = \frac{r_h}{2}\left(1 + \frac{8\pi P}{3} r_h^2\right), \qquad T = \frac{1}{4\pi}\left(r_h^{-1} + 8\pi P, r_h\right), \qquad S = \pi r_h^2, \qquad V = \frac{4\pi}{3} r_h^3, \tag{9.13}$$

and the relations (9.6)-(9.10) reduce to the familiar first law $dM = T, dS + V, dP$ and Smarr identity $M = 2TS - 2PV$. The Hawking-Page transition then follows from the sign change of $G = M - TS$ at the standard minimum of $T(r_h)$ [24], while finite b shifts G by $-, T, b \ln \mathcal{A}_h$ without affecting the structural identities (9.10)-(9.11). The log correction is consistent with one-loop and entanglement computations [6,7], and the classical $f(R)$ Schwarzschild extension providing (9.4) is detailed in [14].

10 Thermodynamic Diagnostics: Visual Evidence

To complement the closed-form expressions for enthalpy $M(r_h)$, temperature $T(r_h)$, entropy $S(r_h)$, Gibbs free energy $G(r_h) = M(r_h) - T(r_h)S(r_h)$, and isobaric heat capacity

$C_P(r_h) = T\,(dS/dr_h)/(dT/dr_h)$ at fixed pressure P and deformation α, we present three diagnostic plots that make the phase structure transparent. Figure 1 displays G versus T using r_h as a parameter and overlays several values of the logarithmic-correction coefficient b. In the uncharged, non-rotating ensemble, the large-black-hole (BH) branch eventually dominates (lowest G) after the minimum temperature is passed. The additive correction $-\,b\,T\ln(4\pi r_h^2/A_0)$ lowers (raises) the free energy for $b > 0$ ($b < 0$), thereby shifting the Hawking-Page (HP) crossing against thermal AdS to *lower* (*higher*) temperatures, in line with [6,7,14,16,24]. Figure 2 plots $C_P(r_h)$ and exhibits poles where $dT/dr_h = 0$. These mark second-order transition lines that separate locally stable ($C_P > 0$) and unstable ($C_P < 0$) segments. While the pole locations are controlled by dT/dr_h and hence are independent of b at leading order, the factor $dS/dr_h = 2\pi r_h + 2\pi\alpha/r_h^3 + 2b/r_h$ makes C_P larger (smaller) on small-r_h branches for $b > 0$ ($b < 0$), effectively enlarging (shrinking) any stability pocket seeded by $\alpha > 0$. Finally, Figure 3 compares $T(r_h)$ with and without the $f(R)$ deformation. At $\alpha = 0$ the Schwarzschild-AdS curve has the standard minimum temperature at $r_h = (8\pi P)^{-1/2}$ [24]. A positive α adds a $+15\alpha/r_h^6$ contribution to dT/dr_h, shifting extrema toward smaller r_h and slightly broadening the range over which a small-BH branch can be locally stable. All three figures use representative, dimensionless choices $(P, \alpha, A_0) = (3 \times 10^{-3}, 10^{-3}, 1)$ purely for qualitative illustration. Any fixed-$P > 0$ slice shows the same patterns, with absolute scales recovered by reinstating G and L as appropriate [16].

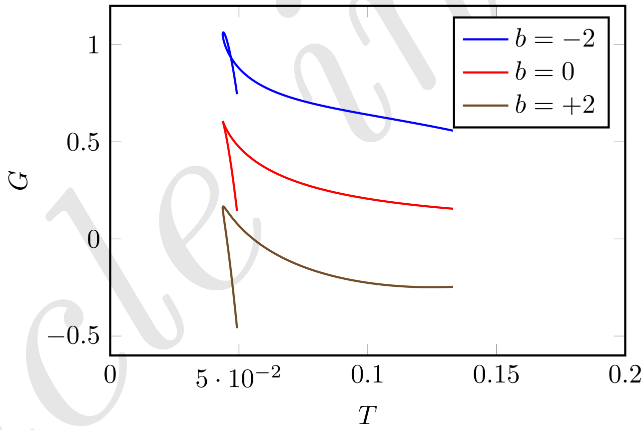

Figure 1: Parametric $G(T)$ at fixed P for several b. Increasing b lowers G at a given T on the large-BH branch and shifts the HP crossing to lower T. Curves are illustrative (dimensionless units).

11 Discussion

The logarithmic correction parameter b encapsulates universal quantum fluctuations around the classical $f(R)$ black-hole background and, being dimensionless, alters thermodynamics in a particularly transparent way. At fixed (P, α) it enters the Gibbs free energy as $\Delta G_b = -T\,b\ln(4\pi r_h^2/A_0)$, thereby shifting the Hawking-Page crossing temperature and reweighting the small-radius branch without modifying the leading geometric relations that follow from the classical equations of motion and the Noether-charge form of entropy [17]. Physically, b aggregates contributions from matter and graviton one-loop determinants as

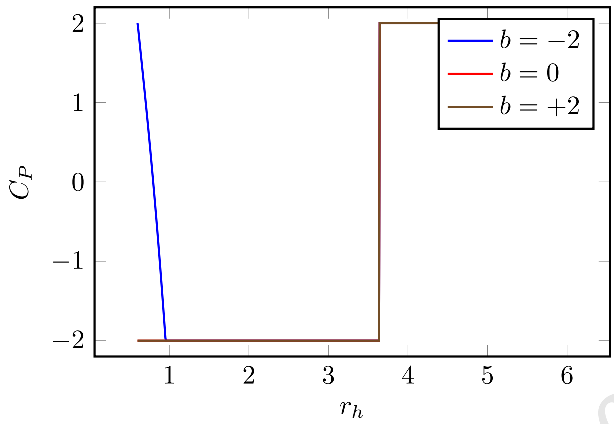

Figure 2: Heat capacity $C_P(r_h)$ at fixed P showing poles where $dT/dr_h = 0$. Positive b raises C_P on small-r_h branches, enlarging local stability pockets seeded by $\alpha > 0$.

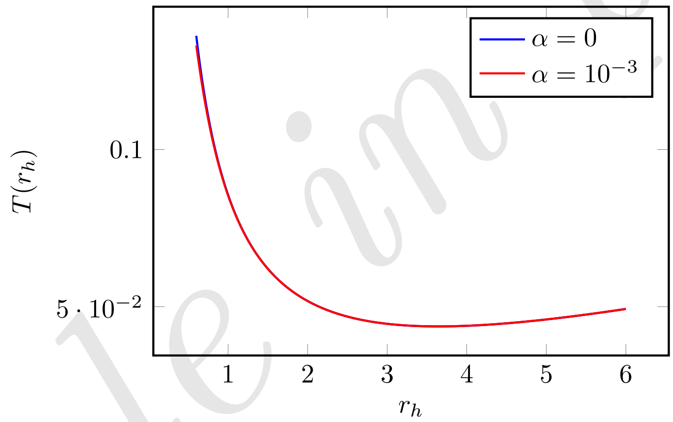

Figure 3: Comparison of $T(r_h)$ with and without the $f(R)$ deformation. The $\alpha = 0$ curve has the standard minimum temperature at $r_h = (8\pi P)^{-1/2}$, $\alpha > 0$ shifts extrema toward smaller r_h.

well as entanglement across the horizon. In Euclidean language it arises from the heat-kernel (or conical) evaluation of functional determinants on the instanton, where the coefficient of $\ln \mathcal{A}_h$ is controlled by the Seeley-DeWitt data and boundary conditions, hence depends on the field content and the choice of ensemble. In the extended phase-space picture, where M is enthalpy and Λ acts as pressure, the scaling structure that yields the generalized Smarr relation is governed by the mass dimensions of (S, P, V, α) and is not sensitive to b, so the dimensionless logarithm leaves Smarr identities intact while still producing observable shifts in $G(T)$ and in heat-capacity sign structure [16]. From a measurability standpoint, a first-principles computation of b can be pursued by a one-loop path-integral on the specific $f(R)$ background used here, including the scalaron sector in the fluctuation operator. Technically this amounts to computing regulated determinants with appropriate boundary terms and then reading off the $\ln \mathcal{A}_h$ coefficient, an approach that also underlies entanglement-entropy derivations of logarithmic terms [6]. Alternatively, in a holographic setting the same coefficient can be inferred from $1/N$ corrections in the dual field theory free energy, offering a route to quantify b indirectly via strongly coupled plasmas where black-hole thermodynamics is mapped to deconfined phases. Phenomenologically, fitting the Hawking-Page temperature and the small-black-hole stability window in the $f(R)$ Schwarzschild extension to high-precision, regulator-independent one-loop calculations on the corresponding solution would convert our treatment into a parameter-free prediction for $G(T)$ and $C_P(T)$ [14]. When compared with other deformations that do change the phase structure already at the classical level most notably electric charge or angular momentum in AdS b plays a subtler role, Reissner-Nordström-AdS and Kerr-AdS backgrounds exhibit van der Waals-type criticality and rich coexistence curves in the (P, T) plane due to additional conserved charges, whereas the logarithmic entropy term primarily displaces transition temperatures and modifies local stability boundaries without introducing new classical critical points to leading order it is expected to leave mean-field critical exponents intact while renormalizing nonuniversal amplitudes and the loci of spinodal lines, a pattern consistent with its origin as a subleading quantum correction rather than a new macroscopic control parameter. In short, the coefficient b provides a clean, UV-sensitive yet thermodynamically sharp dial that shifts the free-energy landscape built from the $f(R)$ extension, and its definitive value should be fixed either by a controlled one-loop computation on the relevant Euclidean saddle or, in holographic realizations, by matching subleading large-N corrections of the boundary free energy to the bulk $\ln \mathcal{A}_h$ term. Recent work suggests that Planck-scale logical and computational limits impose structural constraints on fundamental theory. Information-theoretic incompleteness [25], undecidability [26], Gödel-type restrictions [27], and Tarski's undefinability [28], echoed in third-quantized analyses of string and group field theories [29], motivate asking whether black-hole thermodynamics is similarly constrained, i.e. in its entropy corrections, stability windows, or fluctuation spectra. Testing these meta-logical bounds in the thermodynamic sector is a timely next step.

12 Outlook

A concrete next step is to compute the coefficient b from first principles by evaluating one-loop determinants on the Euclidean $f(R)$ Schwarzschild background, treating the scalaron that renders $f(R)$ ghost-free as an additional propagating spin-0 mode coupled to curvature and incorporating the standard spin-2 graviton and Faddeev-Popov ghosts. Technically this entails fixing a covariant gauge (e.g., de Donder), constructing the quadratic fluctuation operator about the smooth conical-defect-free saddle at inverse temperature β, and extracting the logarithmic term in the one-loop partition function $\ln Z_{\text{1-loop}} \sim -\frac{1}{2}\sum_s (-1)^{F_s} \ln \det \Delta_s$, whose $\ln(\mathcal{A}_h)$ coefficient is set by the integrated Seeley-DeWitt coefficient a_2 (in four dimensions) together with carefully treated zero-modes and boundary conditions [6,7,30–32] for nonextremal horizons the conical-defect method relates the entropy correction directly to the heat-kernel expansion on the smooth cone, while for near- or exactly extremal limits the quantum entropy function provides a cross-check in the near-horizon AdS_2 region [7], and in both routes renormalization-scheme dependence cancels in the entropy so that the resulting b is universal once the field content and boundary conditions are fixed, implementing this program on the specific $f(R)$ Schwarzschild-like solution of Ref. [14] requires the scalaron's effective mass and nonminimal couplings evaluated on that background but is otherwise standard. Beyond neutrality and zero spin, adding Maxwell charge or rotation allows a richer phase diagram already in GR, including van-der-Waals-type criticality in the extended (P, V) ensemble [15,16,33,34], and repeating our analysis with $S \to S_{\text{Wald}} + b \ln \mathcal{A}_h$ on RN-AdS or Kerr-AdS (and then on their $f(R)$ deformations) would quantify how b shifts coexistence lines, spinodal curves, and critical endpoints, because the logarithm is subleading in $\mathcal{A}_h$ one generically expects mean-field critical exponents $(\alpha, \beta, \gamma, \delta) = (0, \frac{1}{2}, 1, 3)$ to remain unchanged while amplitudes and crossover scales receive b-dependent shifts, although multiplicative logarithmic corrections to scaling are plausible near criticality where correlation-length-like quantities diverge, and this can be tested by expanding the Gibbs free energy near the critical point and checking whether the b term introduces nonanalyticities strong enough to alter the Landau functional's universality class [16,35,36]. Finally, it would be valuable to compare ensembles (canonical vs grand-canonical for charge/ang. momentum) and boundary choices (global AdS vs finite cavity) since the spectrum of zero-modes and boundary heat-kernel coefficients, hence b, can change [31,32], and to verify Smarr/first-law consistency at one-loop so that the dimensionless nature of b indeed leaves the Smarr weights intact while modifying $G = M - TS$ in a way that is directly inferable from the determinants.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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