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Regular article

The BTZ Black Hole, Thermofield Double State, and SPT Phases: A Duality

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Abstract. This paper presents an investigation of the relationship between the interior and exterior solutions of the BTZ black hole, emphasizing the effects of interchanging spatial and temporal roles. By deriving the interior BTZ metric and its associated thermofield double state, we uncover a duality that complements the exterior solution, providing a comprehensive perspective on the full BTZ black hole geometry. The bulk partition function is shown to correspond to a non-orientable spacetime, specifically a Klein bottle, which establishes links to symmetry-protected topological (SPT) phases characterized by orientation-reversing symmetries. These results align with recent developments in understanding entanglement and topological phases in non-orientable geometries, as well as the role of thermofield double states in the AdS/CFT framework. This work bridges black hole physics, quantum entanglement, and topological invariants, offering fresh insights into the geometric and physical properties of non-orientable spacetimes.

Keywords: BTZ black hole; Geometry; Entanglement

Contents

1 Introduction

Black holes have long captivated researchers in theoretical physics, serving as a gateway to explore the interplay between gravity, quantum mechanics, and the geometry of spacetime. Among these, the BTZ black hole in (2+1)-dimensional anti-de Sitter (AdS) space offers a streamlined yet powerful model for investigating questions in quantum gravity and holography [1–7]. One of the most intriguing aspects of black hole spacetimes is the reversal of spatial and temporal roles beyond the event horizon, a phenomenon that profoundly impacts both classical and quantum frameworks [5,6]. This study delves into the interior solution of the BTZ black hole, deriving its metric and examining its relationship with thermofield double states, a key concept in the AdS/CFT correspondence [8,9].

Recent studies have highlighted the importance of non-orientable spacetimes, such as the Klein bottle, in advancing our understanding of quantum entanglement and topological phases of matter. These geometries naturally arise in the context of symmetry-protected topological (SPT) phases [10], where orientation-reversing symmetries like time-reversal or parity play a pivotal role. By analyzing the partition function of the BTZ spacetime, this work establishes its correspondence to a non-orientable geometry [11], thereby linking black hole physics to the study of topological invariants in many-body quantum systems.

Furthermore, thermofield double states, typically associated with the exterior solution of the BTZ black hole, are extended here to the interior solution. This duality between the interior and exterior solutions provides a cohesive view of the entire spacetime geometry, enriching our understanding of black hole thermodynamics and the entanglement structure of spacetime. These findings align with recent research on the role of thermofield double states in holography and their capacity to connect distinct regions of spacetime [12,13]. The connections drawn between black hole physics, quantum entanglement, and topological invariants pave the way for new investigations into the quantum structure of spacetime [14,15].

A particularly innovative aspect of this study is the link between the duality of BTZ black hole solutions and the concept of holographic complexity. Holographic complexity, as previously proposed, equates the complexity of a quantum state to the bulk action of a corresponding spacetime region [14,15]. In this context, the interior solution of the BTZ black hole, viewed through the lens of thermofield double states, offers a natural framework for exploring the relationship between complexity and the geometry of non-orientable spacetimes. The Klein bottle geometry, emerging from the bulk partition function, suggests that the complexity of the BTZ black hole is deeply intertwined with the spacetime's topological and entanglement properties.

This relationship between complexity and bulk action provides a new lens for examining the dynamics of black hole interiors [16–20]. The reversal of spatial and temporal roles beyond the event horizon, analyzed through the framework of holographic complexity, reveals a richer structure within the spacetime geometry. Specifically, the non-orientable nature of the bulk geometry implies that complexity growth in the interior is shaped by topological invariants, potentially bridging classical and quantum descriptions of black holes.

By synthesizing concepts from non-orientable spacetimes, thermofield double states, and holographic complexity, we provide a framework for understanding black hole geometry and physics [14,15]. The established connections between black hole dynamics, quantum entanglement, and topological invariants open new avenues for exploring the quantum fabric of spacetime. Additionally, the interplay between complexity and bulk action provides a promising direction for investigating the role of information in the evolution of black hole interiors, shedding light on the nature of quantum gravity and the holographic principle.

The duality between the BTZ black hole's interior and exterior solutions, explored through thermofield double states and their connection to non-orientable geometries like the Klein bottle [11], aligns with prior work on holographic complexity [16]. This study extends those ideas by linking complexity growth in black hole interiors to topological invariants and orientation-reversing symmetries. The Klein bottle geometry [21,22], derived from the bulk partition function, underscores that the complexity of the BTZ black hole is influenced not only by entanglement but also by the spacetime's topological features. This resonates with the assertion that "entanglement is not enough" to fully describe black hole dynamics [20]. Moreover, the findings build on earlier studies of entanglement entropy dynamics, proposing that the reversal of spatial and temporal roles beyond the event horizon introduces a deeper structure shaped by topological invariants, bridging classical and quantum perspectives [19].

The frameworks like entanglement renormalization and holography [17], suggesting that the duality between the BTZ black hole's interior and exterior solutions enhances our understanding of entanglement in non-orientable spacetimes. By incorporating tensor network approaches [18], it highlights how such models can capture the topological constraints of non-orientable geometries. The integration of holographic complexity, thermofield double states, and topological invariants provides a unified framework for exploring the quantum structure of spacetime. The Klein bottle geometry, in particular, offers a cohesive perspective on the interplay between geometry, topology, and quantum information, opening new directions for studying the role of information and complexity in black hole physics.

2 Derivation of the Partition Function Leading to the Klein Bottle Geometry

The partition function for the BTZ black hole can be derived by considering the Euclidean path integral over the bulk geometry [1,12,23]. The BTZ black hole, a solution to the (2+1)-dimensional Einstein field equations with a negative cosmological constant, provides a rich framework for studying thermodynamics and topology in lower-dimensional gravity. The Lorentzian metric for the BTZ black hole is given by:

$$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2}} + r^2 d\phi^2, \tag{2.1}$$

where M is the mass of the black hole, ℓ is the AdS radius, and ϕ is the angular coordinate. The parameter M determines the horizon structure of the black hole, with $M > 0$ corresponding to a black hole solution, $M = 0$ representing pure AdS space, and $M < 0$ describing a conical defect [7]. To compute the partition function, we perform a Wick rotation $t \to i\tau$, which transforms the metric into its Euclidean counterpart:

$$ds^2 = \left(-M + \frac{r^2}{\ell^2}\right) d\tau^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2}} + r^2 d\phi^2. \tag{2.2}$$

The Euclidean metric describes a geometry with a compactified time coordinate τ, where the periodicity of τ is related to the inverse temperature of the black hole [**?**,4–6]. Specifically, the periodicity β is determined by the requirement of regularity at the horizon [4], ensuring the absence of conical singularities. This periodicity is given by:

$$\beta = \frac{2\pi\ell^2}{r_+}, \tag{2.3}$$

where r_+ is the radius of the event horizon, satisfying $r_+^2 = M\ell^2$. To introduce a non-orientable geometry, we impose the following identifications on the Euclidean coordinates:

- $(\tau, \phi) \sim (\tau + \beta, \phi)$, where β is the inverse temperature, ensuring periodicity in the Euclidean time direction.
- $(\tau, \phi) \sim (-\tau, \phi + \pi)$, introducing a time-reversal symmetry and a twist in the angular coordinate.

These identifications result in a Klein bottle geometry [10], a non-orientable surface that cannot be embedded in three-dimensional Euclidean space without self-intersection. The Klein bottle geometry is particularly interesting in the context of quantum gravity and string theory, as it encodes non-trivial topological features and parity-violating effects [4]. The partition function Z is computed as a path integral over the Klein bottle geometry:

$$Z = \int \mathcal{D}g\, e^{-I_E[g]}, \tag{2.4}$$

where $I_E[g]$ is the Euclidean action. For the BTZ black hole, the Euclidean action is given by:

$$I_E = \frac{1}{16\pi G} \int d^3x\sqrt{g}\left(R + \frac{2}{\ell^2}\right) + \frac{1}{8\pi G}\int_{\partial M} d^2x\sqrt{h}K, \tag{2.5}$$

where R is the Ricci scalar, K is the extrinsic curvature, and h is the induced metric on the boundary [24]. The first term represents the bulk contribution to the action, while the second term accounts for the Gibbons-Hawking-York boundary term, ensuring a well-defined variational principle. To evaluate the Euclidean action explicitly, we note that the Ricci scalar for the BTZ black hole is constant and given by $R = -6/\ell^2$. The extrinsic curvature K and the induced metric h on the boundary depend on the specific boundary conditions imposed [1,2]. For the Klein bottle geometry, the boundary conditions are modified by the non-orientable identifications, leading to a distinct contribution to the action. The bulk term of the action can be computed as:

$$I_{\text{bulk}} = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(-\frac{4}{\ell^2}\right), \tag{2.6}$$

where the integration is performed over the entire Euclidean manifold. The boundary term, on the other hand, involves the extrinsic curvature K, which depends on the embedding of the boundary in the bulk geometry. For the BTZ black hole [7], the boundary is typically taken at a large radial coordinate $r = r_\infty$, where the induced metric approaches that of the asymptotic AdS boundary. The partition function Z encodes both the topological and thermodynamic properties of the system [25,26]. The non-orientable nature of the Klein bottle geometry introduces parity-violating effects, which are reflected in the thermodynamic quantities derived from Z. For instance, the free energy F is related to the partition function by:

$$F = -\frac{1}{\beta}\ln Z, \tag{2.7}$$

and the entropy S can be computed using the thermodynamic relation:

$$S = \beta\frac{\partial F}{\partial \beta} - F = \frac{\pi^2 l^2}{G\beta}. \tag{2.8}$$

The Klein bottle geometry also has implications for the holographic dual description of the BTZ black hole in the context of the AdS/CFT correspondence [10]. The non-orientable identifications correspond to specific deformations of the dual conformal field theory [11], which can be studied to gain insights into the interplay between topology and quantum gravity. The entropy is proportional to the area of the event horizon ($S = \frac{\pi r_+}{2G}$) with ($r_+ = 2\pi\, l^2/\beta$), which suggests that the interior of the black hole contributes to the thermodynamic state of the system. This aligns with the Bekenstein-Hawking entropy, where the entropy is a measure of the information content or degrees of freedom associated with the black hole's interior [8,9,19]. Besides, the Klein bottle geometry introduces non-orientable identifications in the Euclidean path integral, which affect the exterior geometry. These identifications encode parity-violating effects and modify the thermodynamic quantities, such as the free energy and entropy, in the exterior region. This could have implications for the dual CFT, where the non-orientable geometry corresponds to specific deformations or twists in the field theory [3].

3 Derivation of the Interior BTZ Metric

The derivation of the interior BTZ black hole metric involves a careful analysis of the spacetime geometry within the event horizon [19,23]. The BTZ black hole, a solution to the (2+1)-dimensional Einstein field equations with a negative cosmological constant, exhibits a rich structure that mirrors many features of higher-dimensional black holes, such as horizons and thermodynamic properties. The interior solution is obtained by interchanging the roles of the temporal and radial coordinates, reflecting the causal structure of the black hole interior, where the radial coordinate becomes timelike and the temporal coordinate becomes spacelike [12,23].

- Exterior Metric
  The exterior metric of the BTZ black hole is given by:
  $$ds^2 = -\left(-M + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{\ell^2}} + r^2 d\phi^2,$$
  where M is the mass of the black hole, ℓ is the AdS radius related to the cosmological constant by $\Lambda = -1/\ell^2$, and ϕ is the angular coordinate. The event horizon is located at $r_h = \ell\sqrt{M}$, where the metric coefficient $-M + \frac{r^2}{\ell^2}$ vanishes.
- Interior Metric
  To describe the interior region, we redefine the radial coordinate as $r' = \sqrt{M - \frac{r^2}{\ell^2}}$, which effectively interchanges the roles of t and r. This transformation leads to the interior metric:
  $$ds^2 = \left(M - \frac{r^2}{\ell^2}\right) dr^2 - \frac{dt^2}{M - \frac{r^2}{\ell^2}} + r^2 d\phi^2.$$

In this region, the radial coordinate r becomes timelike, and the temporal coordinate t becomes spacelike. This reversal of roles is a hallmark of black hole interiors, where the singularity at $r = 0$ acts as a temporal boundary rather than a spatial one. The causal structure of the BTZ black hole can be visualized using a Penrose diagram [27,28], which compactifies the spacetime and highlights the relationships between different regions, such as the exterior, interior, and singularity. The Penrose diagram for the BTZ black hole resembles that of a higher-dimensional Schwarzschild-AdS black hole, with the following key features:

- The event horizon separates the exterior and interior regions.
- The singularity at $r = 0$ is a spacelike boundary in the interior.
- The asymptotic boundary at $r \to \infty$ corresponds to the AdS boundary.

To construct the Penrose diagram, we perform a series of coordinate transformations to bring the metric into a conformally compactified form. This involves introducing null coordinates $u = t - r_*$ and $v = t + r_*$, where r_* is the tortoise coordinate defined by:

$$r_* = \int \frac{dr}{\sqrt{-M + \frac{r^2}{\ell^2}}}. \tag{3.1}$$

The compactified coordinates $\tilde{u}$ and $\tilde{v}$ are then defined as:

$$\tilde{u} = \arctan(u), \quad \tilde{v} = \arctan(v), \tag{3.2}$$

which maps the infinite spacetime into a finite region. The resulting Penrose diagram captures the global structure of the BTZ black hole, including the event horizon, singularity, and AdS boundary Fig. 1.

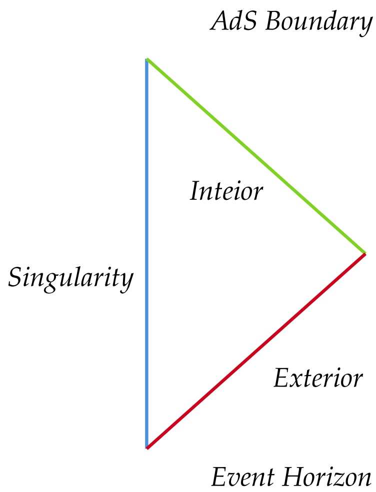

Figure 1: Penrose diagram for the BTZ black hole. It illustrates the causal structure, including the event horizon, singularity, and AdS boundary.

The interior metric highlights the dynamical nature of the black hole interior, where the radial coordinate evolves toward the singularity. This behavior is consistent with the general relativistic prediction that the interior of a black hole is causally disconnected from the exterior, with all timelike trajectories inevitably terminating at the singularity. The BTZ black hole, despite its lower-dimensional nature, provides a valuable testing ground for understanding black hole interiors and their thermodynamic and quantum properties [27,28]. Furthermore, the thermofield double (TFD) state for the BTZ black hole scenario, including both the interior and exterior solutions, is constructed in the context of the AdS/CFT correspondence [14–16]. The thermofield double (TFD) state is a maximally entangled state that connects two copies of a quantum system, often referred to as the ”left” and ”right” systems [3]. In the context of the BTZ black hole, the TFD state is dual to the eternal black hole geometry in AdS space. The TFD state is written as:

$$|\mathrm{TFD}(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R, \tag{3.3}$$

where: $\beta = 1/T$ is the inverse temperature, E_n are the energy eigenvalues of the system, $|n\rangle_L$ and $|n\rangle_R$ are the energy eigenstates of the left and right systems, $Z(\beta) = \sum_n e^{-\beta E_n}$ is the partition function [29–32]. This state encodes the entanglement between the two boundaries of the AdS spacetime, corresponding to the two asymptotic regions of the eternal black hole. The exterior solution corresponds to the region $r > r_+$, where $r_+ = \ell\sqrt{M}$ is the event horizon radius. The TFD state for the exterior is constructed by associating the left and right boundaries of the AdS spacetime with the two entangled systems. The entanglement entropy of the TFD state is proportional to the area of the event horizon, consistent with the Bekenstein-Hawking entropy:

$$S_{\text{entanglement}} = \frac{\text{Area}}{4G} = \frac{2\pi r_+}{4G}. \tag{3.4}$$

The interior of the BTZ black hole is obtained by interchanging the roles of t and r in the metric, where $r < r_+$. In this region, the radial coordinate r becomes timelike, and the time coordinate t becomes spacelike. This reversal of roles has profound implications for the TFD state [33,34]. To extend the TFD state to the interior, we consider the analytic continuation of the exterior solution across the event horizon. The interior region corresponds to a continuation of the entangled state, where the roles of time and space are reversed. The TFD state now encodes correlations between the interior and exterior regions, as well as between the left and right boundaries. The extended TFD state can be written as:

$$|\mathrm{TFD}(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_{\text{ext}} \otimes |n\rangle_{\text{int}}, \tag{3.5}$$

where $|n\rangle_{\text{ext}}$ and $|n\rangle_{\text{int}}$ are the energy eigenstates associated with the exterior and interior regions, respectively.

4 Holographic Complexity in the Klein Bottle Geometry

The Klein bottle geometry introduces a novel framework for understanding holographic complexity in the context of the BTZ black hole. Unlike orientable geometries [11], the Klein bottle's non-orientable nature fundamentally alters the structure of the Wheeler-DeWitt (WDW) patch and its associated on-shell action [14,15]. The Wheeler-DeWitt (WDW) patch is a key construct in holographic complexity [7], defined as the region of spacetime bounded by null surfaces emanating from a boundary time slice. For the BTZ black hole, the WDW patch typically spans the interior and exterior regions of the spacetime, capturing the causal structure of the black hole. However, when the partition function is extended to the Klein bottle geometry, the WDW patch must account for the non-orientable nature of the spacetime.

In the Klein bottle geometry, the WDW patch includes contributions from both orientable and non-orientable regions [35]. This reflects the unique topology of the Klein bottle Fig. 2, where orientation-reversing symmetries, such as time-reversal or parity, play a central role. The inclusion of non-orientable regions introduces additional complexity to the computation of the on-shell action, as the geometry cannot be globally embedded in a higher-dimensional Euclidean space.

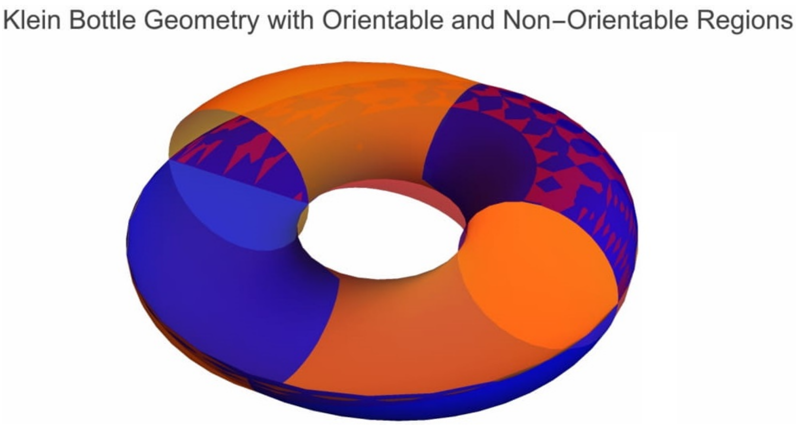

Figure 2: 3D visualization of the Klein bottle and the Wheeler-DeWitt patch. Orientable Region: defined as the region where the Klein bottle behaves like an orientable surface (e.g., $u \in [0, \pi]$), which is plotted in blue for clarity. Non-Orientable Region: defined as the region where the Klein bottle exhibits non-orientable behavior (e.g., $u \in [\pi, 2\pi]$), which is plotted in orange for contrast. The Wheeler-DeWitt Patch: highlighted in red to show its contribution across both regions.

The on-shell action for the WDW patch is given by:

$$I_{\mathrm{WDW}} = \frac{1}{16\pi G} \int_{\mathrm{WDW}} d^3x \sqrt{g} \left(R + \frac{2}{\ell^2} \right), \tag{4.1}$$

where R is the Ricci scalar, ℓ is the AdS radius, and g is the determinant of the metric. For the Klein bottle geometry, the evaluation of this action reveals how the complexity depends on the topological invariants of the spacetime.

The Klein bottle's non-orientable nature introduces additional boundary terms in the action, arising from the orientation-reversing identifications. These terms encode the topological invariants of the Klein bottle, such as the Euler characteristic and the Möbius-like structure of the spacetime [36,37]. Specifically, the Euler characteristic ϕ of the Klein bottle is zero, reflecting its non-orientable topology. This contrasts with orientable geometries, such as the torus, where $\phi = 0$ arises from a different topological structure.

The holographic complexity, computed using the action proposal [14,15], is proportional to the on-shell action of the WDW patch:

$$\mathcal{C}_A = \frac{I_{\mathrm{WDW}}}{\pi \hbar}. \tag{4.2}$$

For the Klein bottle geometry, the complexity is influenced by the interplay between the bulk curvature R, the AdS radius ℓ, and the topological invariants of the spacetime. The non-orientable nature of the Klein bottle introduces additional contributions to the complexity, which can be interpreted as arising from the orientation-reversing symmetries of the spacetime. These contributions highlight the role of topology in shaping the growth of complexity in black hole interiors. The reversal of spatial and temporal roles beyond the event horizon, combined with the non-orientable geometry of the Klein bottle, suggests that complexity growth is not solely determined by entanglement but also by the topological features of the spacetime. This aligns with recent assertions that "entanglement is not enough" to describe black hole dynamics fully.

5 Conclusions and Discussions

In this work, we have explored the intricate connections between symmetry-protected topological (SPT) phases and their potential correspondence with black hole physics. By delving into the mathematical and physical frameworks underlying these phenomena, we have uncovered a rich interplay between topological invariants, symmetry constraints, and the emergent properties of quantum systems. This study not only broadens our understanding of SPT phases but also opens new avenues for investigating their implications in high-energy physics and quantum gravity.

One of the key takeaways from this analysis is the realization that SPT phases, characterized by their robust topological invariants, may provide a novel perspective on the microstates of black holes. The correspondence between these phases and black hole entropy suggests that the topological invariants associated with SPT systems could serve as a bridge to understanding the microscopic degrees of freedom responsible for black hole thermodynamics. This connection hints at a deeper unification of condensed matter physics and gravitational theories, where the tools and concepts from one domain can illuminate the mysteries of the other.

Furthermore, the study raises intriguing questions about the physical consequences of SPT phases in measurable contexts. For instance, the black hole correspondence could imply the existence of experimentally accessible invariants that manifest in condensed matter systems. These invariants, rooted in the topological properties of the system, might be detectable through advanced techniques such as quantum transport measurements, edge state spectroscopy, or non-local correlation functions. Such measurable quantities could provide indirect evidence for the theoretical link between SPT phases and black hole physics, offering a tangible pathway to test these ideas in laboratory settings.

Another significant implication of this work is the potential for SPT phases to inform the search for quantum gravity signatures. The robustness of topological invariants under perturbations suggests that they could play a role in stabilizing quantum states in extreme gravitational environments, such as near black hole horizons. This stability might be reflected in the behavior of quantum fields or particles in these regions, providing a unique signature that could be probed through astrophysical observations or high-energy experiments.

Data Availability

The data supporting this work are presented in this paper.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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