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Regular article

The Effects of Pole Dark Energy on Gravitational Waves

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Abstract. In this paper, we have studied the effects of pole dark energy on the evolution of gravitational waves. The background evolution of gravitational waves in a flat FRW universe is considered, and its dynamics are studied in the presence of pole dark energy. Two different potential functions are considered for the study. Using the field equations, we formulated the perturbed equations governing the evolution of gravitational waves with respect to redshift z within the background of the FRW Universe. Subsequently, we delved into the characteristics of gravitational waves for the pole dark energy model and reached interesting results. We also probed the evolution of the gravitational waves for a universe driven by a cosmological constant and used it as a comparison for the results obtained for pole dark energy. From the analysis, we see that pole dark energy is superior as a dark energy model in driving the spacetime disturbances, compared to a cosmological constant.

Keywords: Pole Dark Energy; Gravitational Waves; Potential; FRW Universe

Contents

1 Introduction

Gravitational waves (GWs), a fascinating consequence of Einstein's general theory of relativity, have been a subject of intrigue for physicists and astronomers alike. In the realm of cosmology and astrophysics, gravitational waves play a pivotal role. These ripples in spacetime carry valuable information about their amplitude, which can be extracted from measurements of cosmic microwave temperature anisotropies and polarization [1,2]. The idea that disturbances in the gravitational field can propagate as waves appears to be intuitive and mirrors the behavior of other types of waves we observe in nature. Einstein demonstrated that gravitational radiation, in the form of gravitational waves, is a natural outcome of his theory, thereby solidifying the connection between gravitational waves and the fundamental principles of general relativity. In the limit of small deviations from Euclidean space-time (or Minkowski space), Einstein's field equations yield a linear wave equation with plane wave solutions. These solutions describe transverse metric perturbations of Minkowski space that travel at the speed of light, exhibiting characteristics analogous to those of electromagnetic waves. While there are several similarities between gravitational and electromagnetic waves, it is essential to note that this comparison has its limitations and should be approached with caution. In the early Universe, gravitons underwent decoupling, leading to the separation of gravitational waves from matter. Consequently, these gravitational waves play a crucial role in constraining and distinguishing cosmological parameters across various cosmological models. Notably, primordial gravitational waves originated from vacuum fluctuations. In a hypothetical scenario during the early Universe, where a seamless transition occurs between an early de Sitter-like phase and a radiation-dominated era, the matter content can be described by a cosmological model that aims to unify the dark sectors of the Universe—namely, dark energy and dark matter. Among all the conceivable models of dark energy in cosmology, gravitational waves can provide valuable comparable insights into epochs when there were fluctuations in the underlying cosmic dynamics. Up to now, a lot of investigations have been done about gravitational waves [3–15].

In physics and astronomy, the first direct discovery of GWs [16] by the LIGO and Virgo collaborations signaled the start of a new area and a new approach to studying the cosmos. The finding of disturbed space-time resulting from the coalescence of two black holes thirty times more massive than the Sun came nearly a century after Einstein first predicted GWs in 1916. During this period, early attempts to discover GWs had failed, and Einstein himself had questioned their very existence at one point [17]. Strong indirect evidence that gravitational waves were released at the rate suggested by the General Theory of Relativity [18] was presented by the Hulse-Taylor binary pulsar [19]. For observations restricted to the audio band, which corresponds to GWs that may be seen at frequencies roughly between 10 and 1000 Hz, there is currently a growing collection of GW detections [21] by the LIGO, Virgo, and KAGRA collaborations. Pulsar timing arrays (PTAs), which are sensitive to GWs at nanohertz (nHz) frequencies, provide an alternative view into the GW-bright Universe. Inspiralling supermassive black hole binaries (SMBHBs) are one possible source of these [22–24]. Other sources include cosmological phase transitions [25], cosmic strings [26], and quantum fluctuations [27] in the early universe. The most theoretically motivated of these are GWs from SMBHBs. Over the years more and more observational data is starting to pour in. Search for GWs with the MeerKAT Pulsar Timing Array (MPTA), is a significant advancement in this direction. This dataset consists of 4.5 years of observations recorded with the MeerKAT L-band receiver (856-1712 MHz) [28,29].

According to recent observations, there is strong evidence that our Universe is currently undergoing accelerated expansion. Clues from various sources, including type Ia supernovae, the Cosmic Microwave Background (CMB), large-scale structure (LSS), and WMAP observations, all point to this cosmic acceleration [20,30–35]. The driving force behind this expansion is an enigmatic substance known as Dark Energy. Dark Energy possesses positive energy density and sufficient negative pressure, which violates the strong energy condition, that is $\omega = p/\rho < -1/3$. Numerous theoretical models have been put forth by researchers to elucidate the enigmatic nature of dark energy [36,37]. Despite these efforts, the true essence of dark energy remains elusive. Among the simplest contenders is the cosmological constant Λ with $\omega = -1$, originally introduced by Einstein to account for the Universe's static behavior. Another straightforward candidate is the quintessence, characterized by a spatially homogeneous scalar field ϕ that gradually descends a potential as $\phi \to \infty$ [38,39]. The scientific literature also features other dark energy candidates, including tachyon, Chaplygin gas, holographic dark energy, dilaton, k-essence, DBI-essence, hessence, and new agegraphic dark energy, all of which play essential roles in driving the Universe's acceleration [40–55].

The presence of a pole in the kinetic term has proven highly effective in the study of inflation within theoretical frameworks. These pole kinetic terms exhibit quantum stability and attractor properties, making them valuable tools. While a scalar field with a pole in its kinetic term is commonly employed for studying cosmological inflation, it can also serve as dark energy, leading to what is known as the pole dark energy model [56,57]. When poles are applied to dark energy theories, they reveal an intriguing connection between thawing and freezing models, as well as the potential for enhanced plateaus characterized by "superattractor-like" behavior. Notably, even simple models incorporating pole dark energy can yield an equation-of-state evolution with $\omega(z) < -0.9$, a feat that would typically be challenging for other potential forms. Furthermore, the kinetic term pole provides an interesting perspective in relation to the swampland criteria for observationally viable dark energy models [58]. Herein, our primary objective is to explore the characteristics of gravitational waves within the context of the pole dark energy model [58] in a Friedmann-Robertson-Walker (FRW) background of the Universe.

2 Background Equations of Gravitational Waves in a Flat FRW Universe

For standard and flat FRW metric

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{2.1}$$

we can obtain Einstein's field equations as (with $8\pi G = c = 1$) [12]

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\left(\rho_m + \rho_D\right), \tag{2.2}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\left(p_m + p_D\right), \tag{2.3}$$

where $a(t)$ is the scale factor, ρ_m is the energy density of dark matter, ρ_D is the energy density of dark energy, and p_m, p_D are the pressures of dark matter and dark energy, respectively.

Given the assumption of separate conservation for dark matter and dark energy, the continuity equations for them are given by

$$\dot{\rho}_m + \frac{3\dot{a}}{a}\left(\rho_m + p_m\right) = 0, \tag{2.4}$$

and

$$\dot{\rho}_D + \frac{3\dot{a}}{a}\left(\rho_D + p_D\right) = 0. \tag{2.5}$$

Considering dark matter satisfies the equation of state $p_m = \omega_m \rho_m$, where ω_m is the constant equation of state parameter, we obtain

$$\rho_m = \rho_{m_0} a^{-3(1+\omega_m)} = \rho_{m_0}\left(1+z\right)^{3(1+\omega_m)}, \tag{2.6}$$

where ρ_{m_0} is the present value of the dark matter density and $z = \frac{a_0}{a} - 1$ is the redshift.

To study the evolution of gravitational waves we will perform a perturbative study of the above equations by putting $\tilde{g}_{\mu\nu} = g_{\mu\nu} + \eta_{\mu\nu}$ in the field equations, where $g_{\mu\nu}$ is the metric tensor and $\eta_{\mu\nu}$ is a small perturbation. We will also add the synchronous condition $\eta_{\mu 0} = 0$. Here we retain the tensorial mode such that $\eta_{\mu\nu} = \eta(t)\Gamma_{\mu\nu}$, where $\Gamma_{\mu\nu}$ is a traceless transverse eigenfunction satisfying the condition $\nabla^2\Gamma_{\mu\nu} = -\xi^2\Gamma_{\mu\nu}$. Here ξ is the wave number times the velocity of light, i.e., $\xi = \frac{2\pi c}{\lambda}$ (λ is the wavelength). The governing equations of the evolution of gravitational waves for a flat universe are

$$\ddot{\eta}(t) - \frac{\dot{a}}{a}\dot{\eta}(t) + \left(\frac{\xi^2}{a^2} - 2\frac{\ddot{a}}{a}\right)\eta(t) = 0, \tag{2.7}$$

which can be written as:

$$\eta^{''}(z) - \left(\frac{\ddot{a}}{\dot{a}^2} - \frac{2}{a}\right)a^2\eta^{'}(z) + \frac{a^4}{\ddot{\eta}^2}\left(\frac{\xi^2}{a^2} - 2\frac{\ddot{a}}{a}\right)\eta(z) = 0, \tag{2.8}$$

where $' \equiv \frac{d}{dz}$.

Next we express the equations (2.2) and (2.3) as follows

$$\frac{\dot{a}^2}{a^2} = H_0^2 X(z) \quad \text{and} \quad \frac{2\ddot{a}}{a} = -H_0^2 Y(z), \tag{2.9}$$

where

$$X(z) = \frac{1}{3H_0^2}\left(\rho_m + \rho_D\right), \tag{2.10}$$

and

$$Y(z) = \frac{1}{3H_0^2}\left[\rho_m\left(1+3\omega_m\right) + \rho_D\left(1+3\omega_c\right)\right], \tag{2.11}$$

where $\omega_m = \frac{p_m}{\rho_m}, \omega_c = \frac{p_D}{\rho_D}$ are the equation of state parameter of dark matter and dark energy respectively, and H_0 is the present value of the Hubble parameter $H = \frac{\dot{a}}{a}$. Exploiting equations (2.9), (2.10) and (2.11), equation (2.8) becomes

$$\eta^{''}(z) + \frac{1}{2(1+z)}\left(\frac{Y(z)}{X(z)} + 4\right)\eta^{'}(z) + \frac{4}{Y^2(z)H_0^4}\left(\xi^2 + \frac{H_0^2 Y(z)}{(1+z)^2}\right)\eta(z) = 0. \tag{2.12}$$

The characteristics of the gravitational waves for the pole dark energy model in flat FRW Universe is going to be investigated in the following sections.

3 Gravitational Waves for Pole Dark Energy Model

Lagrangian for pole dark energy is defined as [58]

$$L_\sigma = -\frac{1}{2}\frac{\kappa}{\sigma^p}(\partial\sigma)^2 - V(\sigma), \tag{3.1}$$

which the pole can be positioned at $\sigma = 0$ without any loss of generality. It possesses a residue of k and an order of p. Poles can emerge in theories due to nonminimal coupling with the gravitational sector, geometric characteristics of the Kähler manifold in supergravity, or as an indication of soft symmetry breaking (as discussed in references [59,60]). In our treatment, we approach it from a phenomenological perspective. Using the transformations

$$\phi = \frac{2\sqrt{k}}{|2-p|}\sigma^{\frac{2-p}{2}}, \qquad \sigma = \left(\frac{|2-p|}{2\sqrt{k}}\right)^{\frac{2}{2-p}} \phi^{\frac{2}{2-p}}, \tag{3.2}$$

we get canonical form the Lagrangian as

$$L_\sigma = -\frac{1}{2}\left(\partial\phi\right)^2 - V(\sigma). \tag{3.3}$$

Using Eq. (3.3), density and pressure in the standard form for FRW universe are respectively expressed as

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{3.4}$$

where dot shows the derivative with respect to time.

For the case $p = 2$, we have

$$\phi = \pm\sqrt{k}\ln\sigma, \qquad \sigma = e^{\pm\frac{\phi}{\sqrt{k}}}. \tag{3.5}$$

Pole dark energy is a field theory model designed to provide $w1$ cosmology and robust attractor behavior. Holographic dark energy (HDE) is motivated by the holographic principle and prescribes a dark-energy density $\rho_{HDE} \propto L^{-2}$ (an IR cutoff L must be chosen). It is a scalar field with a pole in its kinetic term or non-canonical kinetic structure. There can be some deep connections with pole dark energy with holography. A scalar-field model (quintessence, k-essence, tachyon, dilaton, etc.) whose pressure and energy density replicate a certain HDE evolution can be rebuilt. This implies that a certain HDE model can theoretically be used to duplicate the same expansion history using pole dark energy (a particular noncanonical scalar theory). Stated differently, HDE dynamics can be realized in the field using pole dark energy.

Now, from Eq. (3.4), we obtain

$$\dot{\rho}_\phi = \ddot{\phi}\dot{\phi} + V'(\phi)\dot{\phi} = \dot{\phi}\left(\ddot{\phi} + V'(\phi)\right). \tag{3.6}$$

The conservation equation of the canonical form of pole dark energy reads

$$\dot{\rho}_\phi + \frac{3\dot{a}}{a}\left(\rho_\phi + p_\phi\right) = 0. \tag{3.7}$$

Substituting Eqs. (3.4) and (3.5) in Eq. (3.7), we find

$$\dot{\phi}\left(\ddot{\phi} + V'(\phi)\right) + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0. \tag{3.8}$$

Next we take $V(\phi) = \frac{1}{2}v_0\dot{\phi}^2$ and insert in equation (3.8) that yields

$$\dot{\phi} = \phi_0 a^{-\frac{3}{1+v_0}} = \phi_0(1+z)^{\frac{3}{1+v_0}} \ \text{ and } \ V(\phi) = \frac{1}{2}v_0\phi_0^2(1+z)^{\frac{6}{1+v_0}}. \tag{3.9}$$

Inserting the expression of $V(\phi)$ from equation (3.9) in (3.4) we get the energy density and pressure of canonical form of pole dark energy as follows:

$$\rho_\phi = \frac{1}{2}(1+v_0)\phi_0^2(1+z)^{\frac{6}{1+v_0}} \quad \text{and} \quad p_\phi = \frac{1}{2}(1-v_0)\phi_0^2(1+z)^{\frac{6}{1+v_0}}, \tag{3.10}$$

which can be expressed as

$$\rho_\phi = \rho_{\phi 0}(1+z)^{\frac{6}{1+v_0}} \quad \text{and} \quad p_\phi = p_{\phi 0}(1+z)^{\frac{6}{1+v_0}}, \tag{3.11}$$

where $\rho_{\phi 0} = \frac{1}{2}(1+v_0)\phi_0^2$ and $p_{\phi 0} = \frac{1}{2}(1-v_0)\phi_0^2$ are the present values of the energy density and pressure of pole dark energy respectively. Now, we define the dimensionless density parameters as

$$\Omega_{m0} = \frac{\rho_{m0}}{3H_0^2}, \quad \text{and} \quad \Omega_{c0} = \frac{\rho_{D0}}{3H_0^2}, \tag{3.12}$$

where ρ_{m0} and ρ_{D0} are the present values of the energy density of the matter and dark energy respectively.

Using equations (2.6), (3.11) and (3.12) in the field equation (2.2) we get

$$H(z) = H_0\left[\Omega_{m0}(1+z)^{3(1+\omega_m)} + \Omega_{c0}(1+z)^{\frac{6}{1+v_0}}\right]^{\frac{1}{2}}. \tag{3.13}$$

At present epoch from (3.13) can be written as

$$\Omega_{m0} + \Omega_{c0} = 1. \tag{3.14}$$

For canonical form of pole dark energy model the evolution equation (2.12) of gravitational wave is the function of redshift z with observed cosmological parameters $H_0, \omega_m, \Omega_{m0}, \Omega c0$, where $X(z)$, $Y(z)$ can be computed from (2.10) and (2.11) as follows

$$X(z) = \Omega_{m0}(1+z)^{3(1+\omega_m)} + \Omega_{c0}(1+z)^{\frac{6}{1+v_0}}, \tag{3.15}$$

and

$$Y(z) = \Omega_{m0}\left(1+3\omega_m\right)(1+z)^{3(1+\omega_m)} + 2\Omega_{c0}\left(\frac{2-v_0}{1+v_0}\right)(1+z)^{\frac{6}{1+v_0}}. \tag{3.16}$$

We see that the differential equation (2.12) formed after inserting the expressions of $X(z)$ and $Y(z)$ from (3.15) and (3.16), takes a very complicated form. So it is really difficult to generate any closed-form analytical solution to the equation. Nevertheless, we found an analytical solution for a special case when $v_0 = 5$ and $\omega_m = -1/3$. The solution is given by,

$$\begin{aligned} \eta(z) = \text{Differential Root}\Big[&\text{Function}\Big[\{x, y\}, \Big\{\left(-8H_0^2\Omega_{c0} + 8\xi^2 + 8\xi^2 x\right) y[x] \\ &(1+x)^2 H_0^4\Omega_{c0}(3\Omega_{c0} - \Omega_{m0}(1+x))y'[x] + 2(1+x)^3 H_0^4\Omega_{c0}^2 y''[x] = 0, \\ &y[0] = \mathcal{C}_1,\ y'[0] = \mathcal{C}_2\Big\}\Big]\Big][z] \end{aligned} \tag{3.17}$$

Here the 'DifferentialRoot' function represents a holonomic function that satisfies a holonomic differential equation $p_n(x)h^n(x) + p_{n-1}(x)h^{n-1}(x) + ... + p_0(x)h(x) = 0$ with polynomial coefficients $p_i(x)$ and initial values $h^{n-1}(0) = h_{n-1}, ..., h'(0) = h_1, h(0) = h_0$. Moreover $\mathcal{C}_1$ and $\mathcal{C}_2$ represent constants that specify the initial conditions of the universe. To derive a better meaning of the scenario, we plot the wave function $\eta(z)$ against various ranges of redshift z in Figure 1 and Figure 2 for different values of ξ. The other model parameters are taken as $H_0 = 67$, $\omega_m = 0.001$, $v_0 = 1.5$, $\Omega_{m0} = 0.26$, $\Omega_{c0} = 0.74$.

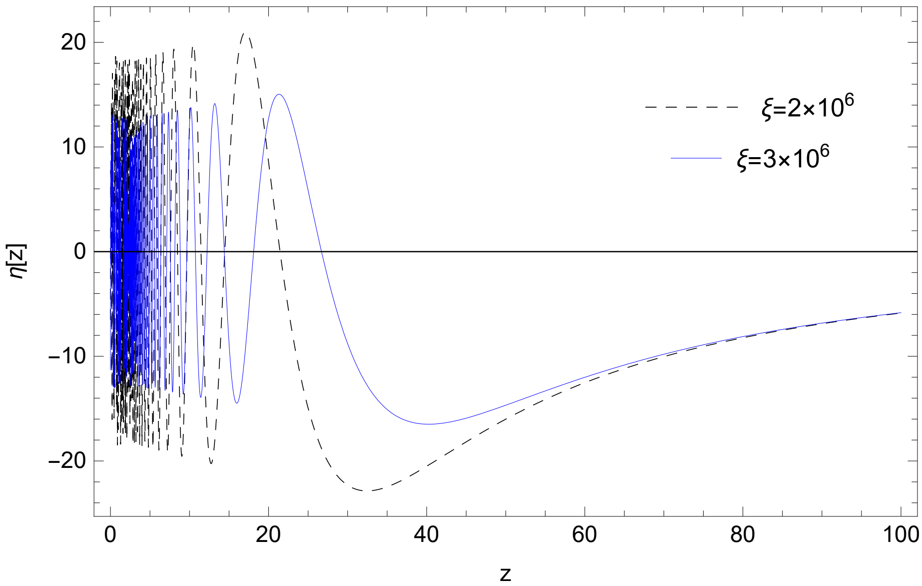

Figure 1: Plot of $\eta(z)$ against redshift $0 \leq z \leq 100$ for different values of ξ. The other parameters are taken as $H_0 = 67$, $\Omega_{m0} = 0.26$, $\Omega_{c0} = 0.74$, $\omega_m = 0.001$, $v_0 = 1.5$.

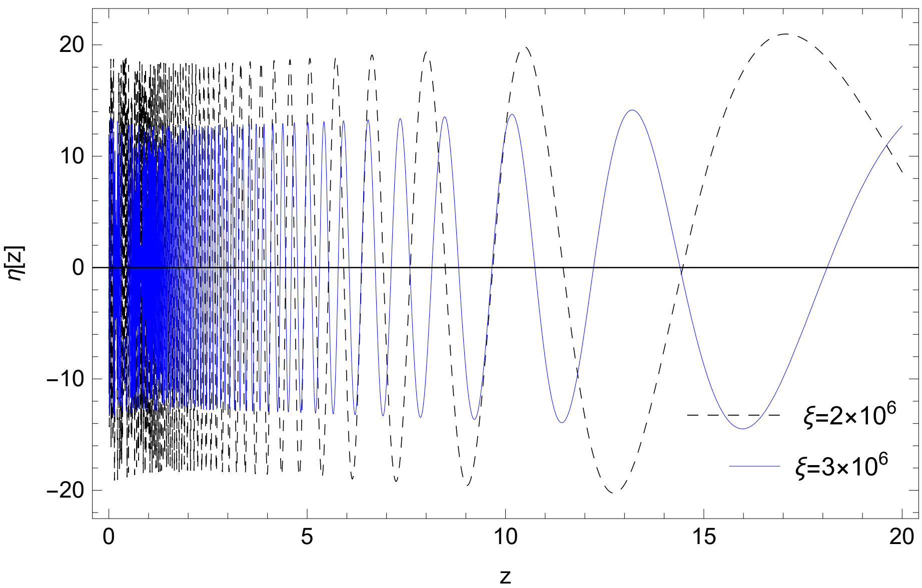

Figure 2: Plot of $\eta(z)$ against redshift $0 \leq z \leq 20$ for different values of ξ to give a magnified image around the present universe ($z = 0$). The other parameters are taken as $H_0 = 67$, $\Omega_{m0} = 0.26$, $\Omega_{c0} = 0.74$, $\omega_m = 0.001$, $v_0 = 1.5$.

3.1 Power law potential

A power law potential can be transformed into the following form

$$V \sim \sigma^n \Rightarrow V \sim \phi^{\frac{2n}{2-p}}, \tag{3.18}$$

which for $p = 1$ becomes $V \sim \sigma^n \Rightarrow V \sim \phi^{2n}$, and for $p = 4$ becomes $V \sim \sigma^n \Rightarrow V \sim \phi^{-n}$. These models, often connected to chaotic inflation, were originally introduced by Andrei Linde in the early 1980s. These forms of potential are mathematically simple and lead to straightforward expressions that can be easily connected with observations. It also allows for the exploration of a wide range of cosmological scenarios by altering the value of the index. Power-law potentials serve as baseline models for testing predictions of quantum gravity or modified gravity theories, effects of reheating, non-Gaussianity, or preheating. These models are also useful in calibrating numerical simulations and providing intuition before tackling complex potentials. These potentials played a major role in developing the inflationary paradigm and hence are widely used in cosmology. Now the derivation of Eq. (3.18) leads to

$$V'(\phi) = \frac{2n}{2-p}\phi^{\frac{2n-2+p}{2-p}}. \tag{3.19}$$

Substituting (3.19) in (3.8) we gain

$$\dot{\phi}\left(\ddot{\phi} + \frac{2n}{2-p}\phi^{\frac{2n-2+p}{2-p}}\right) + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0. \tag{3.20}$$

Considering $\dot{\phi} \neq 0$, we have

$$\frac{\ddot{\phi}}{\dot{\phi}} + \frac{2n}{2-p}\frac{\phi^{\frac{2n-2+p}{2-p}}}{\dot{\phi}} = -\frac{3\dot{a}}{a}. \tag{3.21}$$

Since this equation is not directly solvable, we need to make some transformations. We consider the field in terms of the scale factor in a power law form as

$$\phi(a) = \phi_0 a^m, \tag{3.22}$$

where ϕ_0 and m are constants. So we get,

$$\dot{\phi}(a) = \phi_0 m a^{m-1}\dot{a}, \tag{3.23}$$

and

$$\ddot{\phi}(a) = \phi_0 m\left(a^{m-1}\ddot{a} + (m-1)a^{m-2}\dot{a}^2\right). \tag{3.24}$$

Now putting all these equations in Eq. (3.21), we obtain a differential equation for the scale factor $a(t)$,

$$\ddot{a}(t) + \frac{(m+2)\dot{a}^2}{a(t)} + \frac{2n}{m(2-p)}\phi_0^{\frac{2(n+p-2)}{2-p}}a(t)^{\frac{2mn+2mp-4m-p+2}{2-p}} = 0. \tag{3.25}$$

The solution of Eq. (3.25) is obtained as,

$$a(t) = \mathcal{IF}\left[\frac{\sqrt{m\gamma}\ {}_2F_1\left[\frac{1}{2}, -\frac{(3+m)(-2+p)}{2\gamma}, 1-\frac{(3+m)(-2+p)}{2\gamma}, \frac{2n\phi_0^{-\frac{2(-2+n+p)}{-2+p}}}{m\gamma C_1}\right]\sqrt{\frac{1-2n\phi_0^{-\frac{2(-2+n+p)}{-2+p}}}{m\gamma C_1}}}{(3+m)\sqrt{\phi_0^{\frac{-2(n+p)}{-2+p}}\left(-2n\phi_0^{\frac{4}{-2+p}} + m\gamma\phi_0^{\frac{2(n+p)}{-2+p}}C_1\right)}}\right][t+C_2], \tag{3.26}$$

where '$\mathcal{IF}$' represents InverseFunction and ${}_2F_1$ represents a hypergeometric function. Moreover C_1, C_2 are arbitrary constants and $\gamma = 6 + mn - 3p$.

Putting the above expression for $a(t)$ in Eq. (3.22), we get

$$\phi(t) = \phi_0 \left(\mathcal{IF} \left[\frac{\sqrt{m\gamma}\, {}_2F_1 \left[\frac{1}{2}, -\frac{(3+m)(-2+p)}{2\gamma}, 1 - \frac{(3+m)(-2+p)}{2\gamma}, \frac{2n\phi_0^{-\frac{2(-2+n+p)}{-2+p}}}{m\gamma C_1} \sqrt{\frac{1-2n\phi_0^{-\frac{2(-2+n+p)}{-2+p}}}{m\gamma C_1}} \right]}{(3+m)\sqrt{\phi_0^{\frac{-2(n+p)}{-2+p}} \left(-2n\phi_0^{\frac{4}{-2+p}} + m\gamma\phi_0^{\frac{2(n+p)}{-2+p}} C_1 \right)}} \right] [t + C_2] \right)^m. \tag{3.27}$$

3.2 Exponential potential

For an exponential potential, we have

$$V \sim e^{-\lambda\sigma} \Rightarrow V \sim e^{\frac{-\lambda\sqrt{k}}{\phi}}, \tag{3.28}$$

Exponential potentials appear naturally in many fundamental theories. Kaluza-Klein theories and string theory compactifications often yield exponential terms in the effective potential. Supergravity and brane-world scenarios also lead to exponential forms after dimensional reduction. These models offer a theoretically motivated bridge between fundamental physics and cosmology. Exponential potentials admit scaling solutions where the energy density of the scalar field scales with the background fluid. This means that the scalar field does not dominate too early, and it can track the background energy density (radiation/matter), and later drive acceleration. These solutions are dynamically stable and act as attractors, making them ideal for models like quintessence or assisted inflation. These potentials provide a continuous transition between accelerating and decelerating phases and can serve as a test-bed for modified gravity or brane-world dynamics. From Eq. (3.28) we get

$$V'(\phi) = \frac{\lambda\sqrt{k}}{\phi^2} e^{\frac{-\lambda\sqrt{k}}{\phi}}. \tag{3.29}$$

Now, we put Eq. (3.28) into Eq. (3.7)

$$\dot{\phi}\left(\ddot{\phi} + \frac{\lambda\sqrt{k}}{\phi^2} e^{\frac{-\lambda\sqrt{k}}{\phi}}\right) + 3\frac{\dot{a}}{a}\dot{\phi}^2 = 0. \tag{3.30}$$

With $\dot{\phi} \neq 0$, we have

$$\frac{\ddot{\phi}}{\dot{\phi}} + \lambda\sqrt{k}\frac{\phi^{-2} e^{\frac{-\lambda\sqrt{k}}{\phi}}}{\phi} = -\frac{3\dot{a}}{a}. \tag{3.31}$$

Using the transformation $\phi(a) = \phi_0 a^m$, we obtain

$$\ddot{a}(t) + \frac{(m+2)\dot{a}(t)^2}{a(t)} + \frac{\lambda\sqrt{k}}{m}\phi_0{}^{-3} e^{\frac{-\lambda\sqrt{k}}{\phi_0 a(t)^m}} a(t)^{-3m+1} = 0, \tag{3.32}$$

which leads to the following solution

$$a(t) = \mathcal{IF}\left[\frac{{}_2F_1\left[\frac{1}{2}, \frac{3+m}{6-m}, \frac{9}{6-m}, -\frac{2e^{-\sqrt{k}}\lambda\phi_0 a^{-m}\sqrt{k}\lambda}{(-6+m)m\phi_0^3 C_3}\right]\sqrt{1 + \frac{2e^{-\sqrt{k}}\lambda\phi_0 a^{-m}\sqrt{k}\lambda}{(-6+m)m\phi_0^3 C_3}}}{(3+m)\sqrt{\frac{2e^{-\sqrt{k}}\lambda\phi_0 a^{-m}\sqrt{k}\lambda}{(-6+m)m\phi_0^3} + C_3}} \right][t + C_4], \tag{3.33}$$

where C_3 and C_4 are the constant of integration. Now, substituting Eq. (3.33) in (3.22), we find

$$\phi(t) = \phi_0 \left(\mathcal{IF} \left[\frac{{}_2F_1 \left[\frac{1}{2}, \frac{3+m}{6-m}, \frac{9}{6-m}, -\frac{2e^{-\sqrt{k}}\lambda\phi_0\sqrt{k}\lambda}{(-6+m)m\phi_0^3 C_3} \right] \sqrt{1 + \frac{2e^{-\sqrt{k}}\lambda\phi_0\sqrt{k}\lambda}{(-6+m)m\phi_0^3 C_3}}}{(3+m)\sqrt{\frac{2e^{-\sqrt{k}}\lambda\phi_0\sqrt{k}\lambda}{(-6+m)m\phi_0^3} + C_3}} \right] [t + C_4] \right)^m . \tag{3.34}$$

4 Comparison of Pole dark energy with cosmological constant

In this section, we would like to explore the evolution of gravitational waves in a universe driven by a cosmological constant. This will help us to compare the pole dark energy model with the standard cosmological constant. We expect to get an idea of the efficiency or superiority (if at all) of the pole dark energy model compared to a standard dark energy model. This comparative analysis involving the evolution of gravitational waves will be a novel study in this direction. In the literature, we see that there are some mechanisms to check the efficiency of a dark energy model in an accelerated expanding scenario. The most important of these are the statefinder parameters and the Om diagnostics. The FRW equation with a cosmological constant Λ is given by

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3}\left(\rho_m + \Lambda\right), \tag{4.1}$$

and

$$2\frac{\ddot{a}}{a} = -\frac{1}{3}\left(\rho_m + 3p_m\right) + 2\frac{\Lambda}{3}. \tag{4.2}$$

The continuity equation for Λ can be given by,

$$\dot{\rho}_\Lambda + \frac{3\dot{a}}{a}\left(\rho_\Lambda + p_\Lambda\right) = 0. \tag{4.3}$$

Here the pressure is a negative constant, and the energy density remains constant over time. So ρ_Λ =constant, and the equation of state becomes $p_\Lambda = -\rho_\Lambda$. The density parameter for Λ is taken as

$$\Omega_{\Lambda 0} = \frac{\rho_\Lambda}{3H_0^2}, \tag{4.4}$$

such that $\Omega_{m0} + \Omega_{\Lambda 0} = 1$. Using the above equations we get

$$X(z) = \Omega_{m0}\left(1+z\right)^{3(1+\omega_m)} + \Omega_{\Lambda 0}, \tag{4.5}$$

$$Y(z) = \Omega_{m0}\left(1+3\omega_m\right)\left(1+z\right)^{3(1+\omega_m)} - 2\Omega_{\Lambda 0}. \tag{4.6}$$

Now we will use the above expressions for $X(z)$ and $Y(z)$ in eqn.(2.12) to find the evolution of gravitational wave in this framework. We represent the results in Figs.(3) and (4) in a comparative scenario with the results for the pole dark energy obtained previously. The red-colored plots represent the evolution of gravitational waves in a universe driven by a cosmological constant. In Fig.(3) we have generated the plot for a smaller redshift range $0 < z < 20$ to get a magnified view of the evolution around the present time $z = 0$. From the plot, we see a clear deviation of the waves driven by a cosmological constant from those driven by pole dark energy. Generally, it is seen that the waves in a universe driven by a cosmological constant have less amplitude and are less pronounced than those evolving in a universe driven by pole dark energy. Since we have kept the other parameters of all the plots the same we can take this as a fair comparison between the two models. This definitely gives pole dark energy an upper hand as a dark energy model when compared to a cosmological constant. In Fig.(4) a similar plot has been obtained for a wider range of redshift $0 < z < 50$ to understand the evolution in the past universe. A similar trend is observed in this plot.

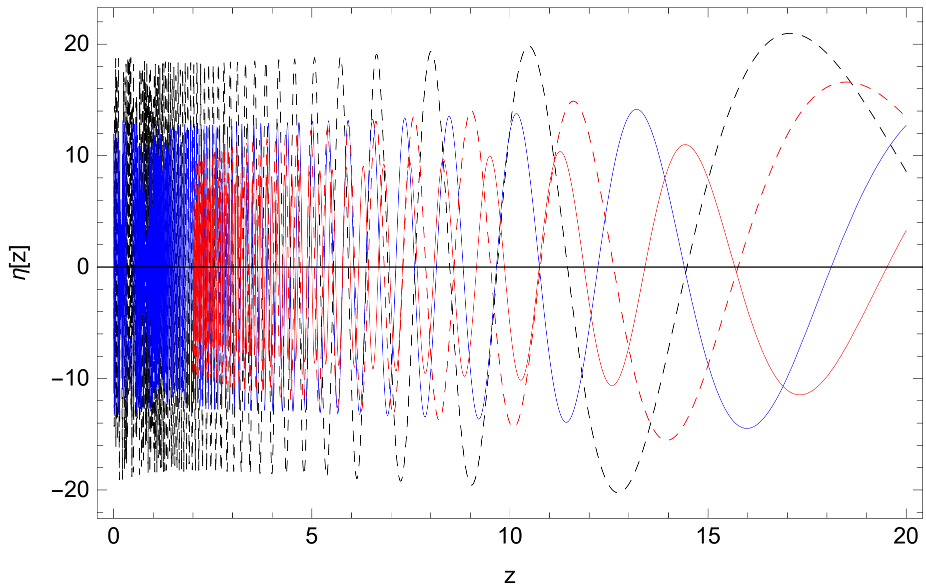

Figure 3: Plot of $\eta(z)$ against redshift $0 \leq z \leq 20$ for both Pole DE and a universe driven by the cosmological constant Λ. A plot of Pole DE for $\xi = 2 \times 10^6$ is given by black dotted lines. Similarly, plot of Pole DE for $\xi = 3 \times 10^6$ is given by blue bold lines. The red dotted plot represents a universe driven by the cosmological constant for $\xi = 2 \times 10^6$. Finally, the red bold line represents a universe driven by the cosmological constant for $\xi = 3 \times 10^6$. The other parameters are taken as $H_0 = 67$, $\Omega_{m0} = 0.26$, $\Omega_{c0} = 0.74$, $\Omega_{\Lambda 0} = 0.74$ $\omega_m = 0.001$, $v_0 = 1.5$.

5 Conclusion

In our study, we investigated the impact of pole dark energy on the evolution of gravitational waves. Our motivation for this study stems from the success of incorporating poles in the kinetic term when studying inflation within theoretical frameworks. These pole kinetic terms demonstrate quantum stability and attractor properties, rendering them valuable tools. In this direction, we explored the flat Friedmann-Robertson-Walker (FRW) model of the Universe, taking into account both dark matter and dark energy. By analyzing separate conservation equations for dark matter and dark energy, we considered energy densities for both components. Then, we explored the properties of gravitational waves in the context of the pole dark energy model within a flat FRW Universe. Through specific transformations, we derived the canonical Lagrangian for pole dark energy. By utilizing this Lagrangian, we obtained expressions for the density and pressure in the standard form applicable to the

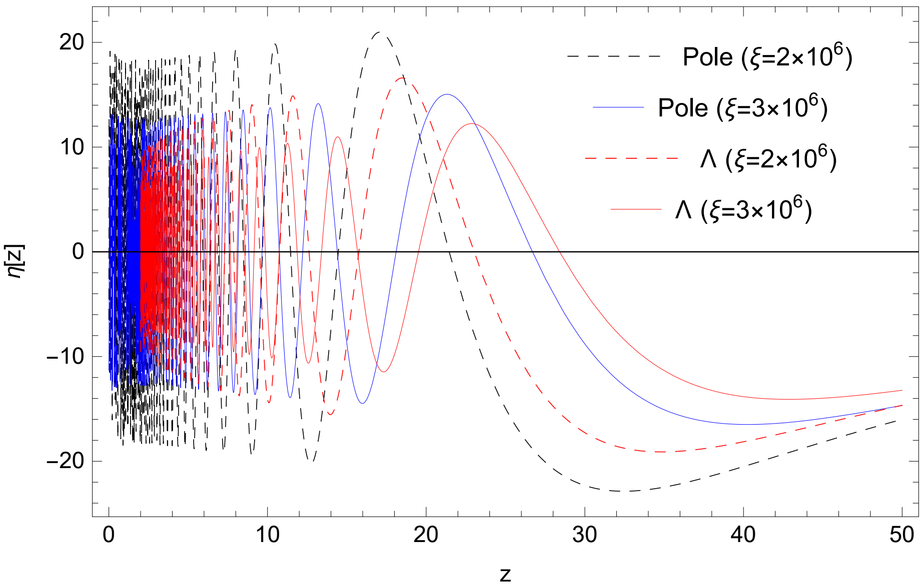

Figure 4: Plot of $\eta(z)$ against redshift $0 \leq z \leq 50$ with the different DE models and different values of ξ. This plot shows the evolution of gravitational waves for a wider range of redshift. The other parameters are taken as $H_0 = 67$, $\Omega_{m0} = 0.26$, $\Omega_{c0} = 0.74$, $\Omega_{\Lambda 0} = 0.74$ $\omega_m = 0.001$, $v_0 = 1.5$.

FRW universe. Using the field equations, we formulated the perturbed equations governing the evolution of gravitational waves with respect to redshift z within the background of the FRW Universe. Subsequently, we delved into the characteristics of gravitational waves for the pole dark energy model. Due to the complexity of the differential equations associated with gravitational waves in this model, we employed graphical analysis to obtain wave curves across various redshift ranges (as depicted in Figs. 1, and 2). We have also explored the evolution of the gravitational waves in a universe driven by a cosmological constant to compare the two results. In Figs. (3) and (4), we have plotted the waves obtained for pole dark energy as well as cosmological constant in a comparative scenario. It is seen that the waves in a universe driven by a cosmological constant have less amplitude and are less pronounced than those evolving in a universe driven by pole dark energy. Since we have kept the other parameters of all the plots the same, we can take this as a fair comparison between the two models. This definitely gives pole dark energy the upper hand as a dark energy model when compared to a cosmological constant. As we see in all figures, the amplitudes increase over time, as $z \to 0$. Moreover, for further investigations, we examined two distinct potential functions in our study and obtained the results.

In future projects, we would like to develop hybrid models using the pole kinetic structures of scalar fields and holographic cutoffs or entropy-derived energy densities. Both frameworks share a thematic overlap as they attempt to address cosmic acceleration with theoretical motivations beyond simple cosmological constants. They both touch on fundamental ideas in theoretical cosmology—field dynamics and quantum gravity. So even if there's no formal connection in the literature, parallels in their goals could inspire hybrid future models. It is worth noting that holographic dark energy (HDE) models, based on the holographic principle, constrain the dark energy density through an infrared cutoff tied to horizon scales. Our pole dark energy framework, with its inherent attractor behavior, could be naturally combined with holographic entropy bounds. Such a hybrid picture might allow the pole kinetic structure to dictate the dynamics of the scalar field [61–64], while holographic cutoffs constrain the overall energy budget. In this sense, the gravitational wave evolution derived here could be further tested against holographic limits on entropy and information content [14,15,65–67]. While we have not carried out a full holographic treatment in this work, the similarities in motivation between HDE and pole dark energy suggest promising avenues for future exploration.

Authors' Contributions

All authors have the same contribution.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The authors declare that there is no conflict of interest.

Ethical Considerations

The authors have diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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