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Analytic Continuation and Temporal Entanglement in Relativistic QFTs Emerging from Quantum Many-Body Systems

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Abstract. We reinterpret the recent prescription for temporal entanglement entropy via analytic continuation in holographic quantum field theories from the vantage point of emergent relativistic quantum field theories (QFTs) arising from quantum many-body systems. By framing this analytic continuation in terms of tensor network constructions and saddle point structures in holography, we identify the operational underpinnings that connect non-relativistic microscopic models to low-energy temporal entanglement phenomena. We provide a physical justification for complex extremal surfaces and elaborate on the non-commutativity of analytic continuation and saddle selection, supporting these insights with analogies to quantum spin chains and Gaussian states. Our analysis reveals that the geometrization of time in strongly correlated many-body systems is not merely formal but possesses physically interpretable manifestations rooted in UV/IR correspondence and tensor network dualities.

Keywords: Quantum Entanglement; Holography; Tensor Networks; Analytic Continuation; Saddle Points; UV/IR Correspondence; Many-Body Systems.

Contents

1 Introduction

Entanglement has emerged as a foundational concept bridging quantum information theory, quantum many-body physics, and the holographic principle in quantum gravity. Initially explored through spatial bipartitions on constant-time slices, entanglement entropy has become a central object in understanding phenomena such as topological phases, quantum phase transitions, renormalization group (RG) flows, and the AdS/CFT correspondence [1–3]. In recent years, the scope of entanglement has expanded beyond static configurations to include entanglement in time, or *temporal entanglement*, which opens new avenues for investigating the quantum structure of spacetime and dynamical correlation patterns in quantum systems.

A particularly intriguing development in this direction is the proposal by Heller, Ori, and Serantes [4] for defining entanglement entropy across timelike separations via analytic continuation. Their construction is formulated in holographic quantum field theories, where extremal surfaces in AdS spacetime are continued into the complexified bulk geometry, providing a novel route to accessing temporal entanglement. This idea resonates with broader efforts to understand the role of time in quantum entanglement, especially in systems where Lorentz invariance emerges only at low energies.

In this work, we reinterpret this analytic continuation prescription through the lens of emergent relativistic QFTs arising from discrete quantum many-body systems. Such systems, ranging from quantum spin chains to tensor network states, often serve as UV-complete models that exhibit approximate Lorentz and conformal invariance at criticality [5]. By studying how temporal entanglement manifests in these settings, we aim to uncover the operational underpinnings of complex extremal surfaces and their connection to coarse-grained descriptions of time-evolved quantum states.

Tensor network approaches provide a particularly fruitful framework for exploring these ideas. Temporal Matrix Product States (tMPS) and other real-time evolution methods encode quantum correlations along the time direction [6,7], naturally introducing non-Hermitian structures and pseudoentropy measures [8]. These constructions bear a deep resemblance to complexified path integrals in holography, where analytically continued geometries govern entanglement across non-spacelike intervals. Moreover, tensor networks such as MERA geometrize the renormalization process, echoing the radial direction of AdS spacetimes and reinforcing the conceptual bridge between tensor architectures and holographic dualities [9,10].

The emergent Lorentz invariance in critical lattice models offers a physical motivation for analytic continuation between space and time. In such systems, spatial bipartitions can, under effective Lorentz boosts, be transformed into temporal partitions, hinting at a unified geometric language for both spatial and temporal entanglement [2,11]. Importantly, this continuation is non-trivial: as emphasized in [4], the analytic continuation and saddle point minimization processes do not generally commute. This non-commutativity has direct analogues in many-body physics, where metastable states, symmetry breaking, and competing low-energy sectors lead to multiple effective saddles in variational or path-integral formulations [12].

By grounding the holographic analytic continuation in the physics of spin chains and time-evolved Gaussian states [13,14], we show that the emergence of complex extremal surfaces is not merely a formal artifact, but a manifestation of real dynamical structures in entangled many-body systems. These insights further suggest that the geometrization of time through tensor networks is tied to physically measurable observables, such as entanglement dynamics and circuit complexity [15].

Our investigation highlights how UV/IR duality, typically understood in terms of scale separation in field theory, also governs the analytic structure of entanglement in time. In lattice models, the small-time limit of entanglement entropy recovers vacuum-like behavior, whereas long-time evolution encodes thermalization and operator spreading [16]. These features reflect in the behavior of extremal surfaces after continuation, where physical expectations and geometric criteria must be jointly satisfied.

In summary, this work provides a many-body reinterpretation of temporal entanglement in holographic QFTs by elucidating its correspondence with tensor network constructions and emergent spacetime symmetries. Our results reinforce the physical legitimacy of complexified entanglement structures and pave the way for exploring temporal entanglement in quantum simulators, non-equilibrium systems, and platforms exhibiting quantum criticality.

2 Analytic Continuation and Many-Body Systems

The proposal of analytic continuation to define temporal entanglement entropy [4] invites reinterpretation through the operational frameworks of quantum many-body systems, especially those that realize emergent relativistic behavior. In holographic theories, the continuation involves deforming extremal surfaces in AdS into complexified geometries to probe entanglement across timelike intervals. To understand how such constructions arise from lattice systems, we consider the analytic structure of correlation functions and entanglement measures in discretized models with Lorentz-invariant low-energy sectors.

Let us denote a quantum many-body ground state $|\Psi\rangle$ in $1+1$ dimensions, defined on a spatial lattice with lattice spacing a, governed by a local Hamiltonian H. Entanglement across a spatial interval $A=[x_1,x_2]$ is quantified via the reduced density matrix $\rho_A = Tr_{\bar{A}}\langle\Psi|\Psi\rangle$, and the von Neumann entropy $S_A = -Tr\rho_A \log\rho_A$ captures the spatial entanglement. The standard replica trick expresses S_A as the analytic continuation

$$S_A = -\lim_{n\to 1}\partial_n Tr\rho_A^n, \tag{2.1}$$

where the quantity $Tr\rho_A^n$ is interpreted geometrically in terms of path integrals on n-sheeted Riemann surfaces.

To investigate temporal entanglement, we rotate the spatial interval A through the light cone via a Wick rotation $x \to it$, leading to a Lorentzian region between events p and q separated by proper time τ. In many-body systems, the counterpart of this procedure is a deformation of the spatial cut into the temporal domain. Instead of evaluating ρ_A on a fixed time slice, we consider two-time reduced density matrices $\rho_{t_1t_2}$, constructed from forward and backward time evolution:

$$\rho_{t_1t_2} = Tr_{\bar{\mathcal{R}}}\left(U(t_2-t_1)\rho_0 U^\dagger(t_2-t_1)\right), \tag{2.2}$$

where $\mathcal{R}$ denotes the region in the Hilbert space supporting observables between t_1 and t_2, and $U(t)=e^{-iHt}$ is the time evolution operator. This construction is closely related to Loschmidt echoes and return amplitudes, with $Tr\rho_{t_1t_2}^n$ admitting a natural representation as a Schwinger-Keldysh contour with replicated forward and backward branches.

Tensor networks, particularly temporal matrix product states (tMPS) [6], offer a computational handle on this structure. A tMPS evolves a product state $|\phi\rangle$ under imaginary or real-time evolution and stores intermediate entanglement via tensors arranged along the temporal axis:

$$|\Psi_T\rangle = \sum_{\{s_t\}} \mathrm{Tr}\left[A^{s_1}(t_1)A^{s_2}(t_2)\cdots A^{s_T}(t_T)\right]|s_1s_2\cdots s_T\rangle. \tag{2.3}$$

The network naturally accommodates analytic continuation $t \to ix$ by deforming the time axis into a complex contour, enabling entanglement measures to interpolate between spatial and temporal regimes.

This setup permits a replica path integral description even in the temporal domain. Consider the second Rényi entropy $S_2 = -\log Tr\rho^2_{t_1t_2}$. Using the path integral formalism, we evaluate

$$Tr\rho^2_{t_1t_2} = \int \mathcal{D}[\phi_1, \phi_2]\, e^{i(S[\phi_1]-S[\phi_2])}, \tag{2.4}$$

subject to cyclic boundary conditions that exchange fields at times t_1 and t_2. After Wick rotation to imaginary time, the exponent becomes $-(S_E[\phi_1] + S_E[\phi_2])$, recovering the standard Euclidean picture. In contrast, in the Lorentzian regime the path integral includes interference phases, which become relevant near the light cone.

These observations motivate defining a complexified entanglement entropy functional $S_E(z)$, where z parametrizes the continuation path in the complexified spacetime. In holography, this corresponds to evaluating the area of complex extremal surfaces anchored to boundary points analytically continued across the light cone. In tensor network models, we propose a functional form

$$S_E(z) = \min_{\mathcal{C}_z} \mathcal{A}[\mathcal{C}_z], \tag{2.5}$$

where $\mathcal{C}_z$ is a geodesic or minimal tensor path through the network, and $\mathcal{A}$ is the corresponding cost function, related to circuit complexity or entanglement flux. We conjecture that under analytic continuation, $\mathcal{A}$ exhibits a saddle-point structure analogous to the holographic bulk area, and that the dominant saddle after continuation reflects the competition between different coarse-grained tensor contractions.

A particularly enlightening case is that of critical 1D chains described by conformal field theory (CFT). In Euclidean CFT, entanglement across an interval $A = [x_1, x_2]$ yields [2]

$$S_A^{(n)} = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left(\frac{x_2 - x_1}{\epsilon}\right), \tag{2.6}$$

where c is the central charge and ϵ is a UV cutoff. Upon analytic continuation $x \to x + it$, the interval becomes timelike or null, and the logarithmic dependence picks up imaginary contributions. The real part of the entropy can be interpreted as the physically observable entanglement across the temporal cut, while the imaginary part encodes coherence effects or modular phases.

Thus, analytic continuation defines a natural extension of entanglement entropy into complexified kinematics, and many-body systems?especially those modeled by tMPS or governed by emergent conformal symmetry?provide a precise platform to probe this extension. In this framework, one may derive a generalized entanglement functional that interpolates between spatial and temporal regimes:

$$S_n(\Delta z) = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left(\frac{\Delta z}{\epsilon}\right), \quad \Delta z = \sqrt{(x_2 - x_1)^2 - (t_2 - t_1)^2 + i\delta}, \tag{2.7}$$

where δ ensures the correct branch cut selection in the complex plane. This formula smoothly connects spatial and temporal entanglement and recovers the correct limits in both regimes.

Figure 1 provides a geometric illustration of the analytic continuation procedure used to define temporal entanglement from spatial intervals. The horizontal axis represents the spatial coordinate x, while the vertical axis denotes time t. A spatial bipartition, initially defined between the endpoints x_1 and x_2 on a constant-time slice $t = 0$, is shown as a dashed black line. To probe entanglement across time, the interval is rotated through the light cone

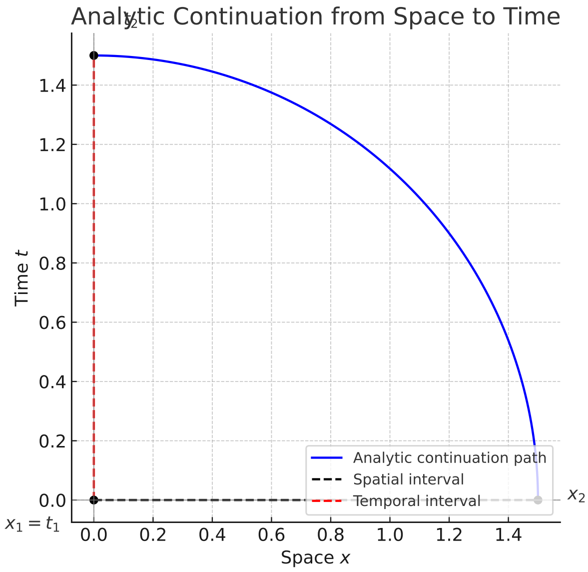

Figure 1: Complexified continuation of an interval from a spatial slice (x axis) to a temporal slice (t axis). The blue arc represents the analytic continuation path in complexified spacetime, mapping a spatial entangling interval to a temporal one.

via analytic continuation, mapping the spatial endpoint x_2 into a point t_2 along the time axis. This transformation is visualized as a curved trajectory in the complexified spacetime plane, represented by the blue arc. The red dashed line marks the final configuration, corresponding to a timelike interval between events t_1 and t_2. This deformation mimics a complex Lorentz boost and captures how real-time evolution entangles regions not separated by space but by time. In the context of tensor networks or holography, such a continuation defines new extremal surfaces or tensor contractions that encode the resulting temporal entanglement structure. The plot highlights the conceptual shift from conventional spatial partitions to genuinely dynamical cuts, central to the framework discussed in this work.

Finally, we emphasize that the analytic continuation is not merely a mathematical trick but reflects operational procedures in quantum simulation. Experimental protocols involving quantum quenches, two-time correlation functions, and temporal mutual information can directly access the structures described above [17], making this approach both physically meaningful and experimentally relevant.

3 Saddle Structures and Non-Commutativity

A central insight of the temporal entanglement framework proposed in [4] is the non-commutativity between analytic continuation and saddle point selection. This subtle phenomenon reflects the intricate structure of path integrals in complexified geometries and has meaningful analogues in quantum many-body systems, particularly in those exhibiting metastability, dynamical phase transitions, or symmetry-breaking sectors. From a technical standpoint, it implies that the order in which one performs analytic continuation and evaluates the dominant contribution to an observable, such as entanglement entropy, can yield qualitatively distinct outcomes.

To formalize this, consider the Euclidean path integral representation of the n-th Rényi entropy, $Tr\rho_A^n$, as a functional integral over fields ϕ defined on an n-sheeted branched manifold:

$$Tr\rho_A^n = \int_{\mathcal{M}_n} \mathcal{D}[\phi]\, e^{-S_E[\phi]}. \tag{3.1}$$

The analytic continuation to Lorentzian signature involves deforming the integration contour and the manifold $\mathcal{M}_n$ into a complexified spacetime. In the semiclassical limit, where $\hbar \to 0$ or the central charge $c \gg 1$, this integral is approximated by a saddle-point expansion:

$$Tr\rho_A^n \approx \sum_i e^{-S_E[\phi_i]}, \tag{3.2}$$

where ϕ_i label classical saddle configurations. Importantly, the analytic continuation modifies both the action $S_E[\phi] \to S[\phi]$ and the boundary conditions, thereby reshaping the entire saddle structure.

If one performs the saddle point evaluation in the Euclidean theory *prior* to continuation, the dominant contribution comes from the minimal real Euclidean action. However, if the continuation is applied first, the relevant extremum may lie off the real slice, and a different saddle with complexified geometry could dominate. This discrepancy arises due to Stokes phenomena in complex analysis, where the steepest descent contours shift discontinuously as parameters are varied [18].

To illustrate this, let us consider a toy model: a scalar field in $1 + 1$ dimensions with effective Euclidean action near a saddle expanded as

$$S_E[\phi] \approx S_0 + \frac{1}{2} \int dx\, \delta\phi(x) \left(-\partial_x^2 + m^2\right) \delta\phi(x), \tag{3.3}$$

where m^2 may become complex under continuation $x \to it$. If $m^2 \to m^2 + i\mu$ for some $\mu \in \mathcal{R}$, then the quadratic form acquires complex eigenvalues, and the path integral becomes dominated by deformed saddles in complex field space. Such configurations, while non-perturbative, can be reliably accessed via Picard-Lefschetz theory [19], which systematically accounts for the contributions of complex saddles connected to the original contour.

In many-body systems, the analog of this phenomenon appears in tensor network path integrals or transfer matrix decompositions. For instance, consider a quantum spin chain at criticality, whose partition function can be expressed via transfer matrices $T(\theta)$ parametrized by an angular deformation θ corresponding to a Wick rotation:

$$Z(\theta) = Tr\left[T(\theta)^L\right], \quad T(\theta) = e^{-\epsilon H + i\theta P}, \tag{3.4}$$

where P is the momentum operator and ϵ is a regularization parameter. Different values of θ interpolate between spatial and temporal evolutions. The spectrum of $T(\theta)$ determines the dominant contribution, and as θ crosses critical lines in the complex plane, level crossings or avoided crossings can occur, signaling a transition between dominant saddles. Such bifurcations correspond to qualitative changes in the entanglement entropy scaling and are direct signatures of the non-commutativity discussed.

Moreover, holographic duals of this structure involve multiple extremal surfaces satisfying the homology constraint but differing in complexified area. Let $\mathcal{A}[\gamma]$ be the generalized area functional for an extremal surface γ, potentially complex-valued. Then the entanglement entropy is given by

$$S = \min_{\gamma} \Re\left(\frac{\mathcal{A}[\gamma]}{4G_N}\right), \tag{3.5}$$

where the minimization is over all admissible complex saddles. The analytic continuation deforms the set of admissible γ, potentially introducing new saddles not visible in the Euclidean sector. The phenomenon where the minimal real part shifts discontinuously across a boundary in parameter space is another manifestation of Stokes phenomena, now in geometric terms.

We propose a general criterion for when saddle non-commutativity becomes physically significant: it occurs when the analytic continuation induces a *bifurcation* in the saddle landscape, such that multiple complex saddles become nearly degenerate in $\Re S[\phi]$. This is analogous to Landau-Ginzburg potentials with metastable minima, and can be detected by analyzing the Hessian of the effective action:

$$\delta^2 S[\phi] \sim \lambda_1 + i\lambda_2, \quad with \Re\lambda_1 \approx 0, \tag{3.6}$$

signaling the emergence of a new competing saddle as a function of the continuation parameter.

In tensor networks, this corresponds to alternative contraction schemes with similar computational cost but different physical interpretations. For instance, in a time-folded tensor network representing $Tr\rho_A^n$, one can deform the contraction contour around the entangling surface to explore different saddle geometries, much like one explores different geodesics in the bulk. The dominance of a given path depends sensitively on the analytic structure of the underlying tensors and the imposed boundary conditions.

Figures 2 and 3 illustrate two complementary aspects of the saddle structure underlying analytic continuation in temporal entanglement calculations.

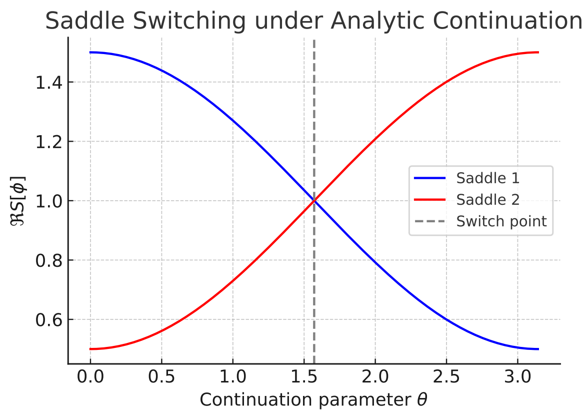

Figure 2: Switching of dominant saddle contributions under analytic continuation. The two candidate saddles exchange dominance at a critical value of the continuation parameter $\theta = \pi/2$, illustrating the non-commutativity between continuation and extremization.

Figure 2 depicts the switching behavior between two competing saddle points as a function of a continuation parameter θ, which can be thought of as an angular deformation interpolating between spatial and temporal cuts. Each curve represents the real part of the action evaluated on a distinct saddle. As the parameter evolves, the originally subdominant saddle becomes energetically favorable, overtaking the initial extremum at a critical point $\theta = \pi/2$. This transition reflects the non-commutativity between analytic continuation and saddle point selection: had one chosen the dominant saddle in the unrotated (Euclidean) theory, the resulting entanglement estimate would differ from that obtained after continuation. Such saddle switching is reminiscent of phase transitions in effective theories and signals the presence of non-analyticities in entanglement observables.

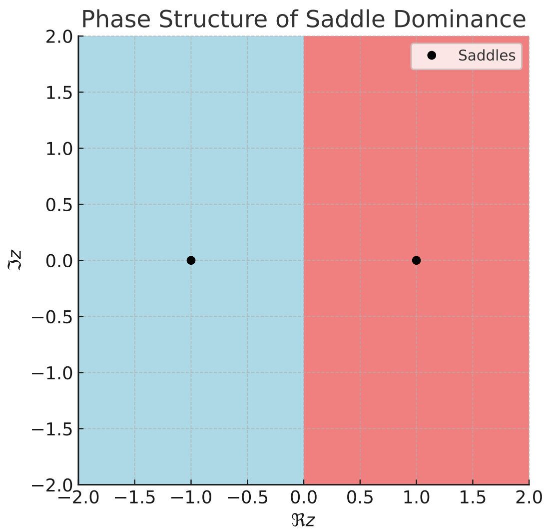

Figure 3: Phase structure of saddle dominance in the complex plane. The shaded regions indicate which saddle dominates as a function of the complexified continuation variable z. The transition boundary marks a non-analyticity in saddle dominance, signaling a Stokes phenomenon.

Figure 3 provides a broader view of the complexified parameter space, displaying the *phase structure* of saddle dominance across the complex plane. Each shaded region indicates the domain where a particular saddle contributes most significantly to the path integral. The boundary between these regions corresponds to a Stokes line, across which the identity of the dominant saddle changes. These saddle bifurcation curves are governed by the relative real parts of the action, and their crossing reflects a topological restructuring of the steepest descent contours. This picture provides a geometric explanation for the sudden change in entanglement behavior upon continuation and helps identify parameter regimes where subtle quantum interference or coherence effects may emerge.

Together, these figures emphasize that analytic continuation is not merely an extrapolation of real-valued quantities but introduces qualitative changes in the entanglement landscape. The transition between saddles and the resulting non-analyticities are not artifacts of approximation but essential features of the underlying quantum structure, potentially observable in controlled quantum simulation experiments and tensor network analyses.

In summary, the non-commutativity of analytic continuation and saddle selection reflects a deep interplay between complex geometry and quantum entanglement structure. Whether viewed through the lens of complex path integrals, tensor network contractions, or holographic extremal surfaces, this non-trivial ordering exposes the layered nature of entanglement dynamics. The emergence of competing saddles after continuation signals phase transitions in the effective information geometry, and their study opens a rich arena for exploring novel quantum phenomena beyond the reach of traditional Euclidean methods.

4 UV-IR Correspondence and Emergent Geometry

The principle of UV-IR correspondence, originally formulated in the context of AdS/CFT [20, 21], posits a duality between short-distance (ultraviolet) degrees of freedom on the boundary and long-distance (infrared) features in the bulk geometry. In the context of temporal entanglement, this principle acquires new significance, as analytic continuation rotates spatial regions into timelike intervals, effectively mapping UV modes localized in space to IR behavior extended across time. We argue that this geometric reinterpretation persists not only in holographic models but also in tensor network realizations of emergent quantum field theories and their temporal evolution.

Let us consider a system governed by a local Hamiltonian H with a characteristic correlation length ξ. In ground states or low-temperature regimes, spatial entanglement entropy S_A for a region of size ℓ typically satisfies an area law $S_A \sim \ell^0$ for gapped systems, or exhibits logarithmic corrections $S_A \sim \log \ell$ in critical theories. Upon analytic continuation to a timelike interval of duration τ, the entanglement across this temporal region reflects the spread of quantum correlations in time. However, due to UV/IR mixing under continuation, small spatial regions do not necessarily map to small-time intervals. Rather, the effective size of the temporal entangling region involves a dynamical rescaling:

$$\tau_{\text{eff}} \sim \frac{\ell^z}{v}, \tag{4.1}$$

where z is the dynamical exponent and v is the effective Lieb-Robinson velocity of the system. For Lorentz-invariant systems ($z = 1$), this yields $\tau_{\text{eff}} \sim \ell$, but for generic many-body systems with anisotropic scaling, this continuation becomes nontrivial.

To elucidate the entanglement structure across time, we examine the mutual information between events at different times:

$$I(t_1 : t_2) = S_{t_1} + S_{t_2} - S_{t_1 \cup t_2}, \tag{4.2}$$

where S_{t_i} denotes the entropy associated with measurement at time t_i, and $S_{t_1 \cup t_2}$ is the joint entropy across the time-separated region. In lattice systems, this quantity probes the extent to which local operators retain coherence across time, serving as a diagnostic for information flow and scrambling. When evaluated via tensor network evolutions, such as time-evolving block decimation (TEBD) or temporal MERA structures, the mutual information acquires a geometrical interpretation: it measures the minimal number of entangling tensors connecting the events along the causal cone, analogous to minimal geodesics in holography.

The key observation is that analytic continuation warps this causal structure. While spatial entanglement probes the boundary of a causal diamond in spacetime, temporal entanglement corresponds to surfaces that traverse the interior. In holographic language, the extremal surfaces associated with temporal intervals dip deeper into the bulk, and their area becomes sensitive to IR data even when the endpoints are UV-localized. This results in a renormalization of the entanglement entropy:

$$S_T(\tau) \sim \frac{c}{3} \log \left(\frac{\tau}{\epsilon} \right) + \alpha \left(\frac{\tau}{\xi} \right)^{\delta} + \cdots , \tag{4.3}$$

where c is the central charge, ϵ the UV cutoff, and the subleading power-law term reflects IR dressing due to coarse-graining effects or emergent hydrodynamics. The exponent δ depends on the universality class of the system. This correction, negligible in short-time limits, becomes dominant in the IR, highlighting the temporal analog of area-law violations.

Tensor networks such as MERA and its temporal analogs naturally encode this behavior. In MERA, entanglement entropy corresponds to the number of disentanglers intersected by a causal cone, and UV/IR correspondence is geometrized by layers in the network. In temporal networks, such as time-folded tensor evolutions or causal influence diagrams, entanglement across time is associated with the complexity of forward-backward tensor contractions. This motivates defining an effective temporal entanglement geometry, whose volume or area functional approximates the complexity of reconstructing time-evolved subsystems.

Inspired by this idea, we propose a variational principle for temporal entanglement based on minimal circuit complexity. Let $C[\mathcal{O}(t)]$ denote the complexity of preparing a time-evolved observable $\mathcal{O}(t)$ from the vacuum. Then the temporal entanglement entropy across $[t_1, t_2]$ satisfies the bound:

$$S_T(t_1, t_2) \leq \min_{\mathcal{C}} \{\log \dim \mathcal{H}_{\mathcal{C}}\} \sim \min_{\gamma} \mathcal{A}[\gamma], \tag{4.4}$$

where $\mathcal{H}_{\mathcal{C}}$ is the Hilbert space accessible through the circuit $\mathcal{C}$, and γ denotes a minimal tensor path in the network geometry. This connects the area of complex extremal surfaces in AdS with the computational cost of encoding temporal correlations in many-body systems.

Finally, we note that the UV/IR correspondence in temporal entanglement naturally leads to a temporal renormalization group, where increasing the temporal resolution reveals finer entanglement structure akin to integrating in UV modes. This observation resonates with recent studies on modular flow, operator growth, and entanglement contour functions [22], and suggests that the continuation across the light cone should be viewed not only as a kinematic deformation but as a renormalization operation in time.

Figures 4 and 5 offer complementary visualizations of the emergent structure of temporal entanglement as understood from both holographic and tensor network perspectives.

Figure 4 illustrates the geometric interpretation of entanglement across a temporal interval in a complexified bulk spacetime. The horizontal axis represents the boundary time coordinate, while the vertical axis denotes an emergent radial (bulk) coordinate associated with IR scales. An extremal surface γ is anchored at two boundary time points t_1 and t_2, and extends into the bulk, probing deeper energy scales as it spans the temporal separation. This curve captures the holographic encoding of temporal entanglement entropy and reflects the non-local nature of temporal correlations. The surface resides within a light cone, marked by gray dashed lines, emphasizing that the entangling region remains causally connected and consistent with Lorentzian kinematics. The geometry suggests that small time intervals probe only near-boundary (UV) structure, while large intervals extend toward the IR regime, reinforcing the temporal form of UV/IR correspondence.

Figure 5 provides an operational realization of this geometry within tensor network architectures, specifically in time-folded tensor evolutions. Here, each rectangle denotes a local tensor acting along the time direction, forming a discrete analogue of real-time path integrals. The blue-outlined boxes mark the causal region contributing to the entanglement between t_1 and t_2, effectively defining a causal cone. This region contains the entangling tensors that contract through both forward and backward branches of the evolution, and determines the entropic structure through their connectivity. The minimal tensor contraction path through this region is analogous to a geodesic in the bulk, whose effective "area" or "volume" can be related to circuit complexity and entropic cost. This mapping between geometric and computational pictures strengthens the interpretation of entanglement as an emergent spatial and temporal structure arising from underlying quantum circuits.

Together, these figures underscore the deep connection between spacetime geometry, entanglement scaling, and computational representation in quantum many-body systems. They reveal how analytic continuation into the temporal domain gives rise to well-defined,

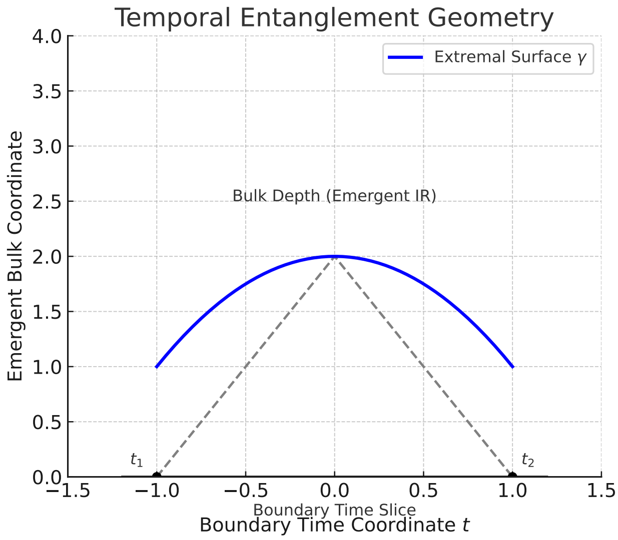

Figure 4: Emergent bulk geometry of temporal entanglement. The extremal surface γ (blue) dips into the IR region of a complexified bulk spacetime, anchored at times t_1 and t_2 on the boundary. This geometry captures the non-local structure of entanglement across time intervals.

physically interpretable structures that mirror their spatial counterparts, while extending the holographic dictionary into dynamical regimes. In summary, temporal entanglement provides a new lens through which the geometry of entanglement and the scaling structure of quantum field theories can be examined. By identifying the correspondence between UV-localized temporal events and IR-dressed geometries?both in holography and tensor networks–we uncover an emergent bulk spacetime interpretation of entanglement in time. This duality enriches the dictionary between quantum information and geometry and opens the door to new analytic and numerical tools for probing dynamics in strongly correlated systems.

5 Conclusion

We have presented a reinterpretation of analytic continuation techniques for temporal entanglement entropy, originally developed in holographic quantum field theories, from the perspective of emergent relativistic QFTs that arise in quantum many-body systems. Our analysis provides a physically grounded framework that connects complex extremal surfaces in AdS geometries to operationally well-defined procedures in lattice systems, tensor networks, and path integral formulations. In particular, we have emphasized how tensor network architectures, especially temporal matrix product states and time-folded evolution schemes, naturally accommodate entanglement across time-like intervals and offer a microscopic basis for understanding holographic continuation.

A key result of our investigation is the non-commutativity between analytic continuation and saddle point selection, which we interpreted through the lens of competing semiclassical configurations, complexified field configurations, and alternative tensor contraction

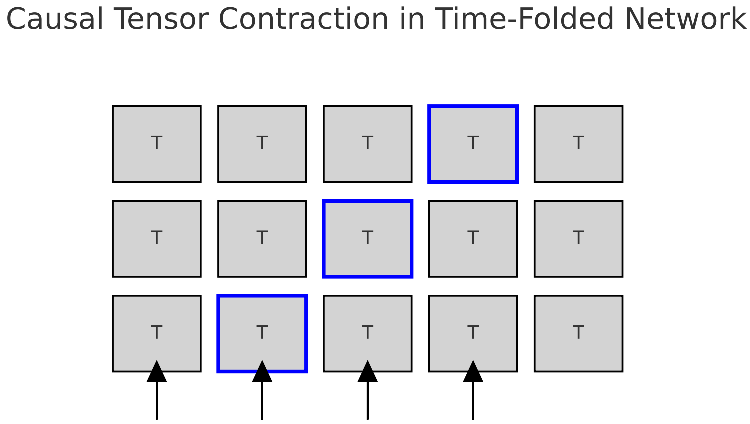

Figure 5: Tensor network representation of time-folded evolution. Each box represents a local tensor contraction along the temporal axis. The blue-outlined tensors form a causal cone enclosing the interval $[t_1, t_2]$, contributing dominantly to the entanglement structure.

paths. By introducing the notion of saddle bifurcation and identifying signatures of Stokes phenomena in both holography and many-body dynamics, we have clarified why temporal entanglement probes genuinely distinct physics from spatial entanglement. Our figures provided visualizations of how these saddle structures evolve, how dominant contributions switch under continuation, and how phase boundaries emerge in complexified parameter space.

Furthermore, we explored the implications of temporal continuation for UV/IR correspondence. We demonstrated that temporal entanglement geometries, whether realized via extremal bulk surfaces or causal tensor networks, obey a dynamical scaling relation that reveals how short-range spatial correlations become long-range temporal entanglement. The emergent geometry arising from analytic continuation was shown to encode not just entropic scaling, but also circuit complexity and causal structure. This led us to propose a variational principle in which the temporal entanglement entropy is bounded by the minimal computational cost to reconstruct time-evolved subsystems, bridging quantum information, field theory, and holography in a novel way.

Our study opens several promising avenues for future research. First, the role of complex saddles in non-relativistic or non-integrable models remains largely unexplored, and numerical tensor network simulations could be used to detect and characterize such saddles via Rényi entropy diagnostics or pseudoentropy probes. Second, generalizing our analytic continuation framework to higher dimensions, or to mixed spacetime cuts, could illuminate how temporal entanglement behaves near quantum critical points or under topological constraints. Third, connections to modular flow and operator spreading suggest that temporal entanglement may offer insights into thermalization, chaos, and complexity growth, particularly in systems with emergent gravitational duals or quantum simulation platforms.

Ultimately, our results reinforce the idea that the analytic continuation of entanglement is not a formal exercise but a physical process, one that reveals new aspects of quantum correlations when viewed through the lens of time. As quantum technologies advance, and real-time measurement of entanglement becomes increasingly feasible, we anticipate that the structures uncovered here will serve as guiding principles for both theoretical exploration and experimental discovery.

Data Availability

The manuscript has no associated data or the data will not be deposited.

Conflicts of Interest

The author declares that there is no conflict of interest.

Ethical Considerations

The author has diligently addressed ethical concerns, such as informed consent, plagiarism, data fabrication, misconduct, falsification, double publication, redundancy, submission, and other related matters.

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References

[1] T. Nishioka, S. Ryu and T. Takayanagi, “Holographic entanglement entropy: an overview”, J. Phys. A **42**, 504008 (2009) DOI:10.1088/1751-8113/42/50/504008.

[2] P. Calabrese, F. H. L. Essler and G. Mussardo, “Introduction to ’Quantum Integrability in Out of Equilibrium Systems”, J. Stat. Mech. 064001 (2016) DOI:10.1088/1742-5468/2016/06/064001.

[3] J. Eisert, M. Cramer and M. B. Plenio, “Colloquium: Area laws for the entanglement entropy”, Rev. Mod. Phys. **82**, 277 (2010) DOI:10.1103/RevModPhys.82.277.

[4] M. Heller, F. Ori and A. Serantes, “Temporal Entanglement from Holographic Entanglement Entropy”, [arXiv:2507.17847] (2025).

[5] G. Vidal, “Class of quantum many-body states that can be efficiently simulated”, Phys. Rev. Lett. **101**, 110501 (2008) DOI:10.1103/PhysRevLett.101.110501.

[6] M. C. Bañuls et al., “Matrix product states for dynamical simulation of infinite chains”, Phys. Rev. Lett. **102**, 240603 (2009) DOI:10.1103/PhysRevLett.102.240603.

[7] J. Haegeman et al., “Time-dependent variational principle for quantum lattices”, Phys. Rev. Lett. **107**, 070601 (2011) DOI:10.1103/PhysRevLett.107.070601.

[8] M. A. Nielsen and I. L. Chuang, “Quantum Computation and Quantum Information”, Cambridge University Press (2000) DOI:10.1017/CBO9780511976667.

[9] B. Swingle, "Entanglement renormalization and holography", Phys. Rev. D **86**, 065007 (2012) DOI:10.1103/PhysRevD.86.065007.

[10] G. Evenbly and G. Vidal, "Tensor network renormalization yields the multi-scale entanglement renormalization ansatz", Phys. Rev. Lett. **115**, 180405 (2015) DOI:10.1103/PhysRevLett.115.180405.

[11] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from AdS/CFT", Phys. Rev. Lett. **96**, 181602 (2006) DOI:10.1103/PhysRevLett.96.181602.

[12] G. Guralnik and Z. Guralnik, "Complexified path integrals and the phases of quantum field theory", Annals of Phys. **325**, 2486 (2010) DOI:10.1016/j.aop.2010.05.004.

[13] I. Peschel, "Calculation of reduced density matrices from correlation functions", J. Phys. A **36**, L205 (2003) DOI:10.1088/0305-4470/36/14/101.

[14] G. Vidal, "Efficient classical simulation of slightly entangled quantum computations", Phys. Rev. Lett. **91**, 147902 (2003) DOI:10.1103/PhysRevLett.91.147902.

[15] R. Jefferson and R. C. Myers, "Circuit complexity in quantum field theory", JHEP **10**, 107 (2017) DOI:10.1007/JHEP10(2017)107.

[16] A. Nahum, J. Ruhman, S. Vijay and J. Haah, "Quantum entanglement growth under random unitary dynamics", Phys. Rev. X **7**, 031016 (2017) DOI:10.1103/PhysRevX.7.031016.

[17] A. Elben et al., "Rényi entropies from random quenches in atomic Hubbard and spin models", Phys. Rev. Lett. **120**, 050406 (2018) DOI:10.1103/PhysRevLett.120.050406.

[18] E. Witten, "Analytic Continuation Of Chern-Simons Theory", AMS/IP Stud. Adv. Math. **50**, 347 (2011) DOI:10.1090/amsip/050/16.

[19] M. Kontsevich and Y. Soibelman, "Stability structures, motivic Donaldson-Thomas invariants and cluster transformations", [arXiv:0811.2435] (2008).

[20] L. Susskind and E. Witten, "The Holographic Bound in Anti-de Sitter Space", [arXiv:hep-th/9805114] (1998).

[21] A. W. Peet, "TASI lectures on black holes in string theory", [arXiv:hep-th/0008241] (2000).

[22] Y. Chen and G. Vidal, "Entanglement contour", J. Stat. Mech. P10011 (2014) DOI:10.1088/1742-5468/2014/10/P10011.